Euclid's *Elements*, encoded as an rrxiv paper

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(translation after Heath, 1908; encoding new, CC-BY-4.0)

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Abstract

We publish a complete, machine-readable rendering of Euclid's *Elements* as an rrxiv paper. All thirteen books are encoded: every definition, postulate, common notion, and proposition is registered as an addressable rrxiv `claim`, and every proof is encoded as a sequence of explicit `depends_on` edges to earlier claims. The encoding produces 465 propositions, 109 definitions, 5 postulates, and 5 common notions, connected by over a thousand `depends_on` edges — the full reasoning DAG of the *Elements* is queryable through the rrxiv API.

The encoding serves three purposes: (i) it dogfoods the rrxiv schema on a finite, dependency-rich corpus that has been studied for two thousand years; (ii) it provides a working reproducibility demonstration — every proposition is provable from claims that the rrxiv graph can enumerate, terminating in the five postulates and five common notions; and (iii) it gives agent harnesses a canonical proof corpus to retrieve over.

Books I, II, and III are written in full Heath-density prose with TikZ figures for the canonical constructions (I.1, I.5, I.32, I.47, II.4, II.11, II.14, III.20, III.31, III.36). Books IV through XIII carry the full statement + dependency-edge DAG with condensed proof sketches; rendering them at Heath density is a long-running editorial project, and PRs at `https://github.com/random-walks/rrxiv-paper-euclid-elements` are welcome. The translation follows Heath (1908, public domain) with light modernisation; the rrxiv encoding is released under CC-BY-4.0.

Postulates

Remark 1 (Postulate 1)**.** To draw a straight line from any point to any point.

Remark 2 (Postulate 2)**.** To produce a finite straight line continuously in a straight line.

Remark 3 (Postulate 3)**.** To describe a circle with any centre and distance.

Remark 4 (Postulate 4)**.** That all right angles are equal to one another.

Remark 5 (Postulate 5 (the parallel postulate))**.** That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

Remark 6 (Common Notion 1)**.** Things which are equal to the same thing are also equal to one another.

Remark 7 (Common Notion 2)**.** If equals be added to equals, the wholes are equal.

Remark 8 (Common Notion 3)**.** If equals be subtracted from equals, the remainders are equal.

Remark 9 (Common Notion 4)**.** Things which coincide with one another are equal to one another.

Remark 10 (Common Notion 5)**.** The whole is greater than the part.

Book I: Definitions

Scope 1 (Definition I.1: point)**.** A point is that which has no part.

Scope 2 (Definition I.2: line)**.** A line is breadthless length.

Scope 3 (Definition I.3: extremities of a line)**.** The extremities of a line are points.

Scope 4 (Definition I.4: straight line)**.** A straight line is a line which lies evenly with the points on itself.

Scope 5 (Definition I.5: surface)**.** A surface is that which has length and breadth only.

Scope 6 (Definition I.6: extremities of a surface)**.** The extremities of a surface are lines.

Scope 7 (Definition I.7: plane surface)**.** A plane surface is a surface which lies evenly with the straight lines on itself.

Scope 8 (Definition I.8: plane angle)**.** A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Scope 9 (Definition I.9: rectilineal angle)**.** And when the lines containing the angle are straight, the angle is called rectilineal.

Scope 10 (Definition I.10: right angle)**.** When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Scope 11 (Definition I.11: obtuse angle)**.** An obtuse angle is an angle greater than a right angle.

Scope 12 (Definition I.12: acute angle)**.** An acute angle is an angle less than a right angle.

Scope 13 (Definition I.13: boundary)**.** A boundary is that which is an extremity of anything.

Scope 14 (Definition I.14: figure)**.** A figure is that which is contained by any boundary or boundaries.

Scope 15 (Definition I.15: circle)**.** A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

Scope 16 (Definition I.16: centre)**.** And the point is called the centre of the circle.

Scope 17 (Definition I.17: diameter)**.** A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Scope 18 (Definition I.18: semicircle)**.** A semicircle is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

Scope 19 (Definition I.19: rectilineal figures)**.** Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Scope 20 (Definition I.20: kinds of trilateral)**.** Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Scope 21 (Definition I.21: kinds of trilateral by angle)**.** Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Scope 22 (Definition I.22: kinds of quadrilateral)**.** Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Scope 23 (Definition I.23: parallel straight lines)**.** Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Scope 24 (Definition II.1: rectangle)**.** Any rectangular parallelogram is said to be contained by the two straight lines containing the right angle.

Scope 25 (Definition II.2: gnomon)**.** And in any parallelogrammic area let any one whatever of the parallelograms about its diameter, with the two complements, be called a gnomon.

Scope 26 (Definition III.1: equal circles)**.** Equal circles are those whose diameters are equal, or whose radii are equal.

Scope 27 (Definition III.2: tangent line)**.** A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle.

Scope 28 (Definition III.3: tangent circles)**.** Circles are said to touch one another which, meeting one another, do not cut one another.

Scope 29 (Definition III.4: equidistant chords)**.** In a circle, straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal.

Scope 30 (Definition III.5: more distant chord)**.** And that straight line is said to be at a greater distance on which the greater perpendicular falls.

Scope 31 (Definition III.6: segment of a circle)**.** A segment of a circle is the figure contained by a straight line and a circumference of a circle.

Scope 32 (Definition III.7: angle of a segment)**.** An angle of a segment is that contained by a straight line and a circumference of a circle.

Scope 33 (Definition III.8: angle in a segment)**.** An angle in a segment is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the base of the segment, is contained by the straight lines so joined.

Scope 34 (Definition III.9: standing on an arc)**.** And, when the straight lines containing the angle cut off an arc, the angle is said to stand upon that arc.

Scope 35 (Definition III.10: sector)**.** A sector of a circle is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the arc cut off by them.

Scope 36 (Definition III.11: similar segments)**.** Similar segments of circles are those which admit equal angles, or in which the angles are equal to one another.

Scope 37 (Definition V.1: part)**.** A magnitude is a part of a magnitude, the less of the greater, when it measures the greater.

Scope 38 (Definition V.2: multiple)**.** The greater is a multiple of the less when it is measured by the less.

Scope 39 (Definition V.3: ratio)**.** A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

Scope 40 (Definition V.4: having a ratio)**.** Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another (the Archimedean property).

Scope 41 (Definition V.5: same ratio (Eudoxean equality of ratios))**.** Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Scope 42 (Definition V.6: proportional)**.** Let magnitudes which have the same ratio be called proportional.

Scope 43 (Definition V.7: greater ratio)**.** When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth.

Scope 44 (Definition V.8: proportion (three terms))**.** A proportion in three terms is the least possible.

Scope 45 (Definition V.9: duplicate ratio)**.** When three magnitudes are proportional, the first is said to have to the third the duplicate ratio of that which it has to the second.

Scope 46 (Definition V.10: triplicate ratio)**.** When four magnitudes are continuously proportional, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, in continual proportion of any number of magnitudes.

Scope 47 (Definition V.11: corresponding magnitudes)**.** Antecedents are said to correspond to antecedents, and consequents to consequents.

Scope 48 (Definition V.12: alternate ratio)**.** Alternate ratio means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

Scope 49 (Definition V.13: inverse ratio)**.** Inverse ratio means taking the consequent as antecedent in relation to the antecedent as consequent.

Scope 50 (Definition V.14: composition of a ratio)**.** Composition of a ratio means taking the antecedent together with the consequent as one in relation to the consequent by itself.

Scope 51 (Definition V.15: separation of a ratio)**.** Separation of a ratio means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

Scope 52 (Definition V.16: conversion of a ratio)**.** Conversion of a ratio means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

Scope 53 (Definition V.17: ratio ex aequali)**.** A ratio ex aequali arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last of the first magnitudes, so is the first to the last of the second magnitudes.

Scope 54 (Definition V.18: perturbed proportion)**.** A perturbed proportion arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.

Scope 55 (Definition VII.1: unit)**.** A unit is that by virtue of which each of the things that exist is called one.

Scope 56 (Definition VII.2: number)**.** A number is a multitude composed of units.

Scope 57 (Definition VII.3: part of a number)**.** A number is a part of a number, the less of the greater, when it measures the greater.

Scope 58 (Definition VII.4: parts)**.** But parts when it does not measure it.

Scope 59 (Definition VII.5: multiple)**.** The greater number is a multiple of the less when it is measured by the less.

Scope 60 (Definition VII.6: even number)**.** An even number is that which is divisible into two equal parts.

Scope 61 (Definition VII.7: odd number)**.** An odd number is that which is not divisible into two equal parts, or that which differs by a unit from an even number.

Scope 62 (Definition VII.8: even-times even)**.** An even-times even number is that which is measured by an even number according to an even number.

Scope 63 (Definition VII.9: even-times odd)**.** An even-times odd number is that which is measured by an even number according to an odd number.

Scope 64 (Definition VII.10: odd-times odd)**.** An odd-times odd number is that which is measured by an odd number according to an odd number.

Scope 65 (Definition VII.11: prime number)**.** A prime number is that which is measured by a unit alone.

Scope 66 (Definition VII.12: relatively prime)**.** Numbers prime to one another are those which are measured by a unit alone as a common measure.

Scope 67 (Definition VII.13: composite number)**.** A composite number is that which is measured by some number.

Scope 68 (Definition VII.14: numbers composite to one another). Numbers composite to one another are those which are measured by some number as a common measure.

Scope 69 (Definition VII.15: multiply). A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

Scope 70 (Definition VII.16: plane number). When two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another.

Scope 71 (Definition VII.17: solid number). When three numbers having multiplied one another make some number, the number so produced is solid, and its sides are the numbers which have multiplied one another.

Scope 72 (Definition VII.18: square number). A square number is equal multiplied by equal, or a number which is contained by two equal numbers.

Scope 73 (Definition VII.19: cube number). A cube number is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

Scope 74 (Definition VII.20: proportional numbers). Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

Scope 75 (Definition VII.21: similar plane and solid numbers). Similar plane and solid numbers are those which have their sides proportional.

Scope 76 (Definition VII.22: perfect number). A perfect number is that which is equal to the sum of its own parts (its proper divisors).

Scope 77 (Definition X.1: commensurable magnitudes). Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

Scope 78 (Definition X.2: commensurable in square). Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure.

Scope 79 (Definition X.3: rational and irrational straight lines). With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length and in square or in square only, rational, but those which are incommensurable with it irrational.

Scope 80 (Definition X.4: rational and irrational areas). And let the square on the assigned straight line be called rational and those areas which are commensurable with it rational, but those which are incommensurable with it irrational, and the straight lines which produce them irrational — that is, in case the areas are squares, the sides themselves; in other cases, the straight lines on which the rectangles equal to the areas would be applied.

Scope 81 (Definition X(2).1: binomial straight line). Given a rational straight line and a binomial, divided into its terms, let the square of the greater term be greater than the square of the lesser by the square of a straight line commensurable in length with the greater. Then if the greater term is commensurable in length with the assigned rational straight line, the whole is called a first binomial.

Scope 82 (Definition X(2).2: second binomial)**.** If the lesser term is commensurable in length with the assigned rational straight line, the whole is called a second binomial.

Scope 83 (Definition X(2).3: third binomial)**.** If neither term is commensurable in length with the assigned rational straight line, the whole is called a third binomial.

Scope 84 (Definition X(2).4: fourth binomial)**.** If the square of the greater term exceeds the square of the lesser by the square of a line incommensurable in length with the greater, and the greater term is commensurable in length with the assigned rational straight line, the whole is called a fourth binomial.

Scope 85 (Definition X(2).5: fifth binomial)**.** If, in the same case, the lesser term is commensurable in length with the assigned rational straight line, the whole is called a fifth binomial.

Scope 86 (Definition X(2).6: sixth binomial)**.** If neither term is commensurable in length with the assigned rational straight line, the whole is called a sixth binomial.

Scope 87 (Definition X(3).1: first apotome)**.** Given a rational straight line and an apotome (i.e. a difference of two rationals commensurable in square only), if the square of the whole is greater than the square of the annex by the square of a straight line commensurable in length with the whole, and the whole is commensurable in length with the assigned rational straight line, the apotome is called a first apotome.

Scope 88 (Definition X(3).2: second apotome)**.** If the annex is commensurable in length with the assigned rational straight line, the apotome is called a second apotome.

Scope 89 (Definition X(3).3: third apotome)**.** If neither the whole nor the annex is commensurable in length with the assigned rational straight line, the apotome is called a third apotome.

Scope 90 (Definition X(3).4: fourth apotome)**.** If the square of the whole exceeds the square of the annex by the square of a straight line incommensurable in length with the whole, and the whole is commensurable in length with the assigned rational straight line, the apotome is called a fourth apotome.

Scope 91 (Definition X(3).5: fifth apotome)**.** If, in the same case, the annex is commensurable in length with the assigned rational straight line, the apotome is called a fifth apotome.

Scope 92 (Definition X(3).6: sixth apotome)**.** If neither the whole nor the annex is commensurable in length with the assigned rational straight line, the apotome is called a sixth apotome.

Scope 93 (Definition XI.1: solid)**.** A solid is that which has length, breadth, and depth.

Scope 94 (Definition XI.2: extremity of a solid)**.** An extremity of a solid is a surface.

Scope 95 (Definition XI.3: line at right angles to a plane)**.** A straight line is at right angles to a plane when it makes right angles with all the straight lines which meet it and are in the plane.

Scope 96 (Definition XI.4: plane at right angles to a plane)**.** A plane is at right angles to a plane when the straight lines drawn in one of the planes at right angles to the common section of the planes are at right angles to the remaining plane.

Scope 97 (Definition XI.5: inclination of a line to a plane)**.** The inclination of a straight line to a plane is, assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane and a straight line joined from the foot of the perpendicular to the extremity of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up.

Scope 98 (Definition XI.6: inclination of plane to plane)**.** The inclination of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to the common section at the same point, one in each of the planes.

Scope 99 (Definition XI.7: similarly inclined planes)**.** A plane is said to be similarly inclined to a plane as another to another when the said angles of the inclinations are equal to one another.

Scope 100 (Definition XI.8: parallel planes)**.** Parallel planes are those which do not meet.

Scope 101 (Definition XI.9: similar solid figures)**.** Similar solid figures are those contained by similar planes equal in multitude.

Scope 102 (Definition XI.10: equal and similar solid figures)**.** Equal and similar solid figures are those contained by similar planes equal in multitude and in magnitude.

Scope 103 (Definition XI.11: solid angle)**.** A solid angle is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines. Otherwise: a solid angle is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

Scope 104 (Definition XI.12: pyramid)**.** A pyramid is a solid figure contained by planes which is constructed from one plane to one point.

Scope 105 (Definition XI.13: prism)**.** A prism is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar, and parallel, while the rest are parallelograms.

Scope 106 (Definition XI.14: sphere)**.** When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere.

Scope 107 (Definition XI.15: axis of a sphere)**.** The axis of the sphere is the straight line which remains fixed and about which the semicircle is turned.

Scope 108 (Definition XI.16: centre of a sphere)**.** The centre of the sphere is the same as that of the semicircle.

Scope 109 (Definition XI.17: diameter of a sphere)**.** A diameter of the sphere is any straight line drawn through the centre and terminated in both directions by the surface of the sphere.

Scope 110 (Definition XI.18: cone)**.** When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone. And if the straight line which remains fixed is equal to the remaining side about the right angle which is carried round, the cone will be right-angled; if less, obtuse-angled; and if greater, acute-angled.

Scope 111 (Definition XI.19: axis of a cone)**.** The axis of the cone is the straight line which remains fixed and about which the triangle is turned.

Scope 112 (Definition XI.20: base of a cone)**.** And the base is the circle described by the straight line which is carried round.

Scope 113 (Definition XI.21: cylinder)**.** When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cylinder.

Scope 114 (Definition XI.22: axis of a cylinder)**.** The axis of the cylinder is the straight line which remains fixed and about which the parallelogram is turned.

Scope 115 (Definition XI.23: bases of a cylinder)**.** The bases are the circles described by the two sides opposite to one another which are carried round.

Scope 116 (Definition XI.24: similar cones and cylinders)**.** Similar cones and cylinders are those in which the axes and the diameters of the bases are proportional.

Scope 117 (Definition XI.25: cube)**.** A cube is a solid figure contained by six equal squares.

Scope 118 (Definition XI.26: octahedron)**.** An octahedron is a solid figure contained by eight equal and equilateral triangles.

Scope 119 (Definition XI.27: icosahedron)**.** An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

Scope 120 (Definition XI.28: dodecahedron)**.** A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

Scope 121 (Definition XIII.1: extreme and mean ratio)**.** A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.

Scope 122 (Definition XIII.2: height of a figure)**.** The height of any figure is the perpendicular drawn from the vertex to the base.

Scope 123 (Definition XIII.3: medial straight line)**.** A medial straight line is the mean proportional between two rational straight lines commensurable in square only.

Scope 124 (Definition XIII.4: minor straight line)**.** A minor straight line is the difference of two straight lines incommensurable in square such that the sum of the squares on them is rational, but the rectangle contained by them is medial.

Scope 125 (Definition XIII.5: composite irrational)**.** A straight line which produces with a rational area a medial whole is the irrational straight line such that the square on it added to a rational area makes the whole medial.

1 Book I — Plane Geometry

Claim 1 (Proposition I.1: Equilateral triangle on a given segment)**.** On a given finite straight line to construct an equilateral triangle.

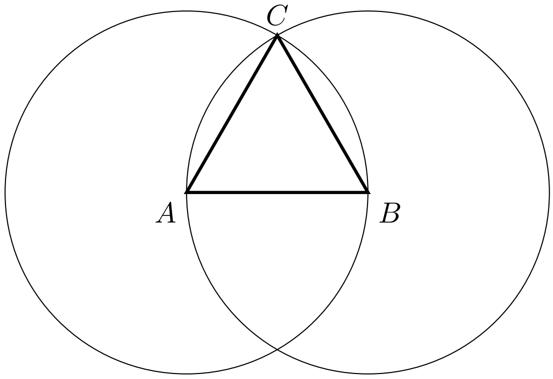

Figure 1: Proposition I.1. The two circles with centres A and B and common radius AB meet at C; the triangle ABC is equilateral.

Evidence 1 (Proof of I.1)**.** Let AB be the given finite straight line. With centre A and distance AB describe the circle BCD (Postulate 3). With centre B and distance BA describe the circle ACE (Postulate 3). From the point C, where the circles cut one another, draw CA and CB (Postulate 1). Since A is the centre of BCD, $AC = AB$ (Definition I.15). Since B is the centre of ACE, $BC = BA$ (Definition I.15). By Common Notion 1, $AC = BC$. Therefore the triangle ABC is equilateral.

Claim 2 (Proposition I.2: Transfer a segment to a given point)**.** To place at a given point (as an extremity) a straight line equal to a given straight line.

Evidence 2 (Proof of I.2)**.** Apply I.1 to obtain an equilateral triangle; produce its sides (Postulate 2); describe a circle with centre at one endpoint of the given segment cutting one of the produced sides; the cut-off equals the given segment by Definition I.15 and Common Notion 1.

Claim 3 (Proposition I.3: Cut off a segment equal to a shorter)**.** Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Evidence 3 (Proof of I.3)**.** Use I.2 to construct, at one extremity of the longer line, a segment equal to the shorter. Then by Definition I.15 the desired cut-off is obtained.

Claim 4 (Proposition I.4: SAS congruence)**.** If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively.

Evidence 4 (Proof of I.4)**.** Superpose one triangle on the other; corresponding sides and angles coincide; by Common Notion 4 the figures are equal.

Claim 5 (Proposition I.5: Pons asinorum)**.** In isosceles triangles the angles at the base are equal to one another; and if the equal straight lines be produced further, the angles under the base will be equal to one another.

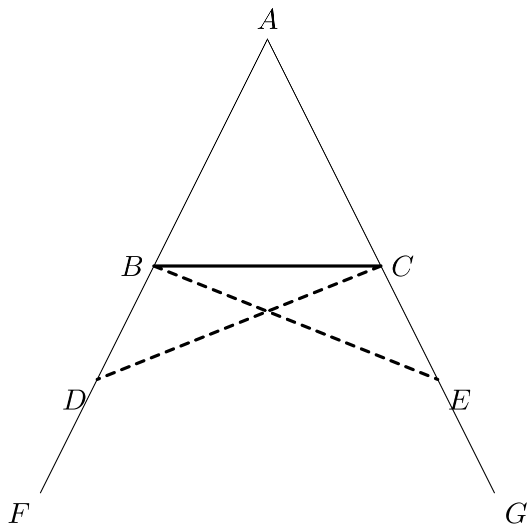

Figure 2: Proposition I.5. With $AB = AC$ and $BD = CE$ on the extensions, triangles ABE and ACD are congruent (SAS, by I.4), whence the base angles $\angle ABC = \angle ACB$.

Evidence 5 (Proof of I.5)**.** Apply I.3 to mark equal segments on the produced sides, then I.4 to two pairs of congruent triangles. Common Notion 3 gives equality of the remaining angles.

Claim 6 (Proposition I.6: Converse of I.5)**.** If in a triangle two angles are equal to one another, the sides which subtend the equal angles will also be equal to one another.

Evidence 6 (Proof of I.6)**.** Suppose the sides unequal; cut off (by I.3) the greater equal to the less; by I.4 the resulting smaller triangle equals the whole, which contradicts Common Notion 5. Therefore the sides are equal.

Claim 7 (Proposition I.7: Uniqueness of triangle on a base)**.** Given two straight lines constructed on a straight line and meeting in a point, there cannot be constructed on the same straight line and on the same side of it two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

Evidence 7 (Proof of I.7)**.** A double application of I.5 yields contradictory angle equalities; by Common Notion 5 the second meeting point cannot exist.

Claim 8 (Proposition I.8: SSS congruence)**.** If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

Evidence 8 (Proof of I.8)**.** Superpose and apply I.7 to rule out a non-coincident image of the contained angle; conclude by Common Notion 4.

Claim 9 (Proposition I.9: Angle bisection)**.** To bisect a given rectilineal angle.

Evidence 9 (Proof of I.9)**.** Apply I.1 to construct an equilateral triangle on a chord between the angle's sides; the line from the vertex to the equilateral's apex bisects the angle by I.8.

Claim 10 (Proposition I.10: Segment bisection)**.** To bisect a given finite straight line.

Evidence 10 (Proof of I.10)**.** Apply I.1 to erect an equilateral triangle on the segment; bisect the opposite angle by I.9; the bisector meets the segment at its midpoint by I.4.

Claim 11 (Proposition I.11: Perpendicular at a point)**.** To draw a straight line at right angles to a given straight line from a given point on it.

Evidence 11 (Proof of I.11)**.** Apply I.3 to mark equal segments either side of the given point, then I.1 to erect an equilateral triangle whose apex line is perpendicular (by I.8 and Definition I.10).

Claim 12 (Proposition I.12: Perpendicular from external point)**.** To draw a perpendicular straight line to a given infinite straight line from a given point not on it.

Evidence 12 (Proof of I.12)**.** With centre at the external point describe a circle (Postulate 3) cutting the given line; bisect the chord by I.10; the line from the external point to the midpoint is perpendicular by I.8.

Claim 13 (Proposition I.13: Adjacent angles sum to two right angles)**.** If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

Evidence 13 (Proof of I.13)**.** Drop a perpendicular by I.11; Common Notions 1–2 sum the resulting two right angles to the unequal-case angle sum.

Claim 14 (Proposition I.14: Converse of I.13)**.** If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

Evidence 14 (Proof of I.14)**.** Suppose not; then I.13 and Common Notion 1 yield contradictory angle sums.

Claim 15 (Proposition I.15: Vertical angles)**.** If two straight lines cut one another, they make the vertical angles equal to one another.

Evidence 15 (Proof of I.15)**.** Apply I.13 twice and Common Notion 3.

Claim 16 (Proposition I.16: Exterior angle of a triangle)**.** In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Evidence 16 (Proof of I.16)**.** Bisect a side by I.10; produce a median; apply I.4 to obtain a congruent triangle that has the interior angle as one of its parts; Common Notion 5 closes the inequality.

Claim 17 (Proposition I.17: Sum of any two angles $<$ two right angles)**.** In any triangle two angles taken together in any manner are less than two right angles.

Evidence 17 (Proof of I.17)**.** Apply I.16 and I.13.

Claim 18 (Proposition I.18: Greater side subtends greater angle)**.** In any triangle the greater side subtends the greater angle.

Evidence 18 (Proof of I.18)**.** Cut off (I.3) a segment on the greater side equal to the lesser; apply I.5 and I.16.

Claim 19 (Proposition I.19: Converse of I.18)**.** In any triangle the greater angle is subtended by the greater side.

Evidence 19 (Proof of I.19)**.** Suppose not; by I.5 and I.18 contradiction.

Claim 20 (Proposition I.20: Triangle inequality)**.** In any triangle two sides taken together in any manner are greater than the remaining one.

Evidence 20 (Proof of I.20)**.** Produce one side to an isosceles configuration by I.3; apply I.5 and I.19.

Claim 21 (Proposition I.21: Interior cevian inequalities)**.** If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

Evidence 21 (Proof of I.21)**.** Two applications of I.20 and one of I.16.

Claim 22 (Proposition I.22: Triangle from three given segments)**.** Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.

Evidence 22 (Proof of I.22)**.** Apply I.20 (the necessity), then I.3 and Postulate 3 (the construction).

Claim 23 (Proposition I.23: Reproduce a given angle)**.** On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.

Evidence 23 (Proof of I.23)**.** Cut equal segments by I.3, construct the matching triangle by I.22, apply I.8.

Claim 24 (Proposition I.24: Hinge inequality)**.** If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

Evidence 24 (Proof of I.24)**.** Apply I.23, I.4, I.5, and I.19.

Claim 25 (Proposition I.25: Converse of I.24)**.** If two triangles have the two sides equal to two sides respectively, but have the one base greater than the other, they will also have the one angle contained by the equal straight lines greater than the other.

Evidence 25 (Proof of I.25)**.** Suppose not and use I.4, I.24.

Claim 26 (Proposition I.26: ASA / AAS congruence)**.** If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely either the side adjoining the equal angles or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle equal to the remaining angle.

Evidence 26 (Proof of I.26)**.** Apply I.4 to the congruent angle–side–angle case; for the AAS case combine I.4 with I.16.

Claim 27 (Proposition I.27: Alternate angles imply parallels)**.** If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

Evidence 27 (Proof of I.27)**.** Suppose the lines meet; I.16 gives a contradiction. By Definition I.23 they are parallel.

Claim 28 (Proposition I.28: Corresponding angles imply parallels)**.** If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

Evidence 28 (Proof of I.28)**.** Reduce to I.27 via I.13 and I.15.

Claim 29 (Proposition I.29: Properties of parallels (uses Postulate 5))**.** A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

Evidence 29 (Proof of I.29)**.** The first use of Postulate 5: any contrary supposition contradicts the parallel postulate via I.13.

Claim 30 (Proposition I.30: Transitivity of parallelism)**.** Straight lines parallel to the same straight line are also parallel to one another.

Evidence 30 (Proof of I.30)**.** Two applications of I.29 and Common Notion 1.

Claim 31 (Proposition I.31: Construct a parallel through a point)**.** Through a given point to draw a straight line parallel to a given straight line.

Evidence 31 (Proof of I.31)**.** Apply I.23 to reproduce the alternate angle at the given point and I.27 to conclude parallelism.

Claim 32 (Proposition I.32: Triangle angle sum)**.** In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

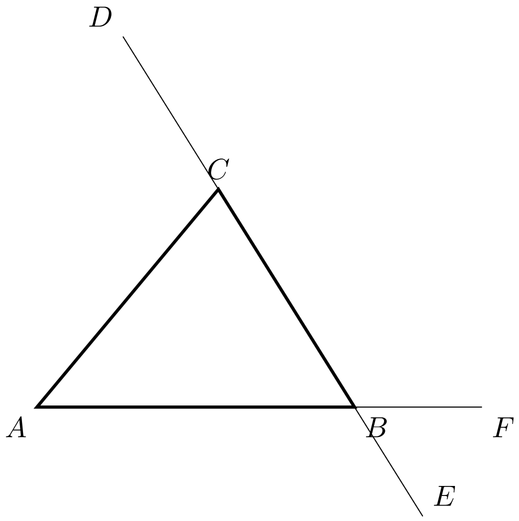

Figure 3: Proposition I.32. Drawing DE through C parallel to AB makes $\angle DCA = \angle CAB$ (alternate, I.29) and $\angle ECB = \angle ABC$ (alternate, I.29); the straight angle at C then sums the three interior angles of $\triangle ABC$ to two right angles.

Evidence 32 (Proof of I.32)**.** Apply I.31 to draw a parallel through the apex; apply I.29 twice and Common Notion 2.

Claim 33 (Proposition I.33: Connecting equal-and-parallel pairs)**.** The straight lines joining equal and parallel straight lines (at the extremities which are in the same directions) are themselves equal and parallel.

Evidence 33 (Proof of I.33)**.** Apply I.4 to the resulting two triangles and I.29 + I.27 for parallelism.

Claim 34 (Proposition I.34: Properties of parallelograms)**.** In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

Evidence 34 (Proof of I.34)**.** Apply I.29 and I.26 to the two triangles cut by the diameter, then Common Notion 2.

Claim 35 (Proposition I.35: Parallelograms with same base & between same parallels)**.** Parallelograms which are on the same base and in the same parallels are equal to one another.

Evidence 35 (Proof of I.35)**.** Apply I.34 and I.4 to congruent triangles; conclude by Common Notions 2–3.

Claim 36 (Proposition I.36: Parallelograms with equal bases)**.** Parallelograms which are on equal bases and in the same parallels are equal to one another.

Evidence 36 (Proof of I.36)**.** Translate one parallelogram via I.33 to share a base with the other, then apply I.35.

Claim 37 (Proposition I.37: Triangles with same base & parallels)**.** Triangles which are on the same base and in the same parallels are equal to one another.

Evidence 37 (Proof of I.37)**.** Complete each triangle to a parallelogram via I.31, then apply I.34 and I.35.

Claim 38 (Proposition I.38: Triangles with equal bases)**.** Triangles which are on equal bases and in the same parallels are equal to one another.

Evidence 38 (Proof of I.38)**.** Same construction as I.37 with I.36 in place of I.35.

Claim 39 (Proposition I.39: Equal triangles on same base ⇒ same parallels)**.** Equal triangles which are on the same base and on the same side are also in the same parallels.

Evidence 39 (Proof of I.39)**.** Apply I.31 to draw a parallel; I.37 forces the second vertex onto it.

Claim 40 (Proposition I.40: Equal triangles on equal bases)**.** Equal triangles which are on equal bases and on the same side are also in the same parallels.

Evidence 40 (Proof of I.40)**.** Same approach as I.39 with I.38 in place of I.37.

Claim 41 (Proposition I.41: Parallelogram is double a triangle)**.** If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

Evidence 41 (Proof of I.41)**.** Apply I.34 and I.37; double via Common Notion 2.

Claim 42 (Proposition I.42: Construct a parallelogram equal in area to a triangle)**.** To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

Evidence 42 (Proof of I.42)**.** Bisect a side by I.10; apply I.23 to set the angle; apply I.31 and I.41.

Claim 43 (Proposition I.43: Complements of a parallelogram)**.** In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

Evidence 43 (Proof of I.43)**.** Apply I.34 to the bisecting diameter and Common Notions 2–3.

Claim 44 (Proposition I.44: Apply a parallelogram to a segment equal to a triangle)**.** To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.

Evidence 44 (Proof of I.44)**.** Apply I.42 and I.43 in sequence.

Claim 45 (Proposition I.45: Apply a parallelogram equal to a rectilineal figure)**.** To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

Evidence 45 (Proof of I.45)**.** Triangulate the figure; sum the parallelograms by repeated I.44 and Common Notion 2.

Claim 46 (Proposition I.46: Construct a square on a segment)**.** On a given straight line to describe a square.

Evidence 46 (Proof of I.46)**.** Apply I.11 to erect perpendiculars; use I.3 to cut off equal segments; use I.31 and I.29 to close the square; verify right angles via I.34.

Claim 47 (Proposition I.47: Pythagoras' theorem)**.** In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

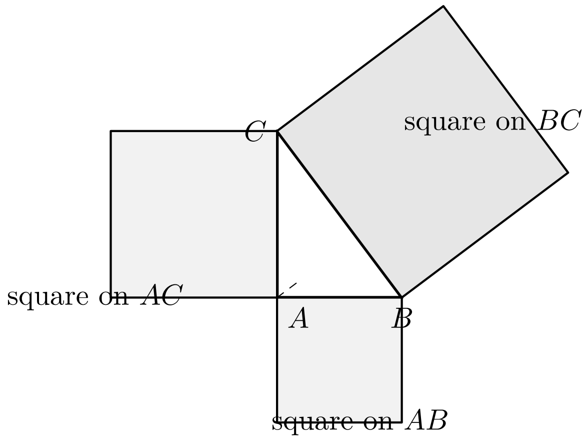

Figure 4: Proposition I.47. The square on the hypotenuse BC is partitioned by the altitude AH extended into two rectangles, each equal (by I.41 + I.46) to a square on a leg; thus $BC^2 = AB^2 + AC^2$.

Evidence 47 (Proof of I.47)**.** Erect squares on each side via I.46; using I.14, I.31 and I.41 show each part-square equals a corresponding parallelogram cut off the hypotenuse-square by the perpendicular from the right angle; sum via Common Notion 2.

Claim 48 (Proposition I.48: Converse of Pythagoras)**.** If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.

Evidence 48 (Proof of I.48)**.** Construct a right triangle with the same two legs by I.11; apply I.47, I.8, and Common Notion 1.

2 Book II — Geometric Algebra

Claim 49 (Proposition II.1: Distributivity of multiplication)**.** If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

Evidence 49 (Proof of II.1)**.** Let A and BC be the two straight lines, and let BC be cut at random at the points D, E. Construct the rectangle contained by A and BC as follows. From B draw BF at right angles to BC (I.11) with BF equal to A. Through F draw FG parallel to BC (I.31), and through C, D, E in turn draw CH, DK, EL parallel to BF (I.31). Then the rectangle BH on the lines A, BC is divided by the parallels DK, EL into the three rectangles BK on A, BD; DL on A, DE; and EH on A, EC (Definition II.1). By Common Notion 2, the whole rectangle equals the sum of these parts: $A \cdot BC = A \cdot BD + A \cdot DE + A \cdot EC$. The argument generalises to any number of cuts on BC.

Claim 50 (Proposition II.2: Whole equals sum of rectangles on parts)**.** If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole.

Evidence 50 (Proof of II.2)**.** Let the straight line AB be cut at random at the point C. Describe on AB the square $ADEB$ (I.46), and through C draw CF parallel to either AD or BE (I.31), so that CF meets DE at F. The square $ADEB$ is thereby divided into two rectangles: $ADFC$ contained by AD and AC (and since $AD = AB$, this rectangle is contained by AB and AC, by Definition II.1), and $CFEB$ contained by CF and CB (again, $CF = AB$, so this rectangle is contained by AB and CB). Their sum (Common Notion 2) is the whole square $ADEB$, which is the square on AB. Therefore the rectangle on AB and AC together with the rectangle on AB and CB equals the square on AB.

Claim 51 (Proposition II.3: Rectangle on a part equals rect on parts plus square)**.** If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment.

Evidence 51 (Proof of II.3)**.** Let AB be cut at C; consider the rectangle on AB, AC. By Proposition II.1, taking AB as the uncut line and the two segments AC, CB of AB itself as the cut line, the rectangle on AB and AC equals the rectangle on AC and AC together with the rectangle on AB and CB — but the rectangle on AC and AC is the square on AC by Definition II.1. Re-expressing the rectangle on AB and CB via II.1 again as the rectangle on AC, CB plus the rectangle on CB, CB (which is the square on CB, again by Definition II.1) and combining, we obtain: the rectangle on AB and AC equals the rectangle on AC and CB plus the square on AC. By Common Notion 2 the equality is preserved when rearranged.

Claim 52 (Proposition II.4: Square of a sum is sum of squares plus twice rectangle)**.** If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

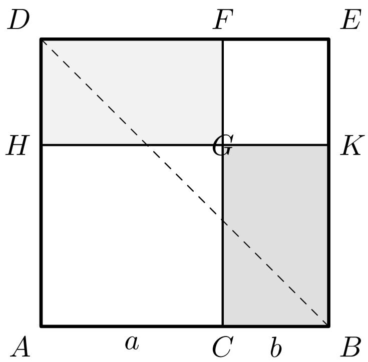

Figure 5: Proposition II.4. The square $ADEB$ on $AB = a + b$ is decomposed by the parallels $CF \parallel AD$ and $HK \parallel AB$ into the square $HDFG$ on $AC = a$, the square $CBKG$ on $CB = b$, and two equal rectangles $AGHD$ and $GFBK$ (by I.43), each equal to $a \cdot b$.

Evidence 52 (Proof of II.4)**.** Let the straight line AB be cut at random at C. Describe on AB the square $ADEB$ (I.46), and draw the diagonal BD. Through C draw CGF parallel to either AD or BE (I.31), meeting BD at G and DE at F. Through G draw HK parallel to either AB or DE (I.31), meeting AD at H and BE at K.

Since CF is parallel to AD and BD falls on them, the exterior angle $\angle BGC$ equals the interior and opposite $\angle BDA$ (I.29). But $\angle BDA = \angle DBA$ since $BA = AD$ (I.5 applied to the isoceles right triangle inside the square). Hence $\angle BGC = \angle GBC$, so $BC = CG$ (I.6), and therefore $CBKG$ is equilateral. Since it has a right angle at B, it is a square on CB (Definition I.22). By the same reasoning $HDFG$ is the square on $HD = AC$.

The complements $AGHD$ and $GFBK$ in the square $ADEB$ are equal rectangles by I.43; each is contained by AC and CB (since $AH = AC$, $HG = CB$, etc.), so each is the rectangle on AC, CB. The four pieces sum to the whole (Common Notion 2):

$$AB^2 \ = \ AC^2 + CB^2 + 2 \cdot (AC \cdot CB),$$

which is $(a + b)^2 = a^2 + 2ab + b^2$ in geometric form.

Claim 53 (Proposition II.5: Rectangle on unequal parts plus square on difference)**.** If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

Evidence 53 (Proof of II.5)**.** Let AB be bisected at C (I.10) and cut unequally at D. Describe on CB the square $CEFB$ (I.46), join BE, and through D draw DG parallel to CE or BF (I.31), meeting BE at H and EF at G. Through H draw KM parallel to AB or EF (I.31), meeting CE at L and BF at M. Through A draw AK parallel to CL or BM (I.31), meeting KM extended at K.

The complement CH equals the complement HF in the square $CEFB$ (I.43). Add to each the square DM; then the rectangle $CDHL$ plus the square $LHMG$ equals the rectangle $DBFG$ plus the same square. But $CDHL$ together with rectangle AC-equivalent piece $AKLC$ (which equals $CDHL$ since $AC = CB$ and the lines are parallel) fills the gnomon NOP, plus the square $LHMG$ on CD, equals the square $CEFB$ on CB. Thus the rectangle $AD \cdot DB$ together with the square on CD equals the square on CB.

Claim 54 (Proposition II.6: Rectangle on bisected-and-produced line)**.** If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line, together with the square on the half, is equal to the square on the straight line made up of the half and the added straight line.

Evidence 54 (Proof of II.6)**.** Let AB be bisected at C (I.10) and produced to D, so that CD is the half plus the added segment BD. Describe on CD the square $CEFD$ (I.46), join DE, and through B draw BG parallel to CE or DF (I.31), meeting DE at H and EF at G. Through H draw KM parallel to AD or EF (I.31), and through A draw AK parallel to CL or DM (I.31).

As in II.5, the complement CH equals the complement HF (I.43). Adding the square $LHMG$ (which is the square on $BC = CB$) to both of $CH + AL$ (the rectangle on AD, DB) shows that the rectangle $AD \cdot DB$ together with the square on CB equals the gnomon plus the small square, which is the square $CEFD$ on CD. Hence $AD \cdot DB + CB^2 = CD^2$ as required.

Claim 55 (Proposition II.7: Squares on whole and segment equal twice rect plus square on remainder)**.** If a straight line be cut at random, the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment together with the square on the remaining segment.

Evidence 55 (Proof of II.7)**.** Let AB be cut at C. Describe on AB the square $ADEB$ (I.46), and through C draw CF parallel to AD or BE (I.31), meeting DE at F. On CB as side construct the square $CBKG$ inside $ADEB$ (a copy of II.4's construction), with HK parallel to AB.

By II.4 the square $ADEB$ on AB equals the square on AC plus the square on CB plus twice the rectangle $AC \cdot CB$. Add the square on CB to both sides of this identity (Common Notion 2):

$$AB^2 + CB^2 \;=\; AC^2 + 2 \cdot CB^2 + 2 \cdot (AC \cdot CB).$$

But $2 \cdot CB^2 + 2 \cdot (AC \cdot CB) = 2 \cdot CB(CB + AC) = 2 \cdot CB \cdot AB$ (Definition II.1; II.1). Hence $AB^2 + CB^2 = 2 \cdot (AB \cdot CB) + AC^2$, as required.

Claim 56 (Proposition II.8: Four-times rectangle plus square on remainder equals square on sum)**.** If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line.

Evidence 56 (Proof of II.8)**.** Let AB be cut at C, and produce AB to D so that $BD = BC$. Then $AD = AB + BC$ and AD is cut at B into the segments AB and $BD = BC$. Apply II.4 to the line AD cut at B: the square on AD equals the squares on AB and BD together with twice the rectangle on AB, BD. Since $BD = BC$, this becomes:

$$AD^2 \;=\; AB^2 + BC^2 + 2 \cdot (AB \cdot BC).$$

Apply II.4 again to AB cut at C, namely $AB^2 = AC^2 + CB^2 + 2 \cdot (AC \cdot CB)$, and substitute. Combining (Common Notion 2) and re-arranging (Common Notion 3) to isolate $4 \cdot (AB \cdot BC)$ on the right side yields:

$$AD^2 \;=\; 4 \cdot (AB \cdot BC) + AC^2,$$

which is $(a + b)^2 = 4ab + (a - b)^2$ in the form Euclid states it.

Claim 57 (Proposition II.9: Squares on unequal segments equal double of two squares)**.** If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.

Evidence 57 (Proof of II.9)**.** Let AB be bisected at C (I.10) and cut unequally at D. At C draw CE at right angles to AB (I.11), with $CE = AC = CB$. Join EA and EB; through D draw DF parallel to CE (I.31), meeting EB at F; through F draw FG parallel to AB (I.31), meeting CE at G. Join AF.

Since $\angle ECA$ is a right angle and $CA = CE$, the triangle ACE is right-isoceles, and $\angle CAE = \angle AEC =$ half a right angle (I.5; I.32). Similarly $\angle CBE = \angle CEB =$ half a right angle. Hence $\angle AEB$ is a right angle (Common Notion 2), and triangle AEB is right-angled at E. By I.47:

$$AB^2 \;=\; AE^2 + EB^2.$$

Now $AE^2 = AC^2 + CE^2 = 2 \cdot AC^2$ (I.47 in $\triangle ACE$, plus $CE = AC$). Similarly inside the right triangles formed by the perpendicular DF at D on AB, I.47 plus the fact that $DF = DE$ (which one shows from the parallel construction) yields, after Common Notions 2 and 3:

$$AD^2 + DB^2 \;=\; 2 \cdot AC^2 + 2 \cdot CD^2.$$

Claim 58 (Proposition II.10: Squares on whole-with-addition and addition equal double of two squares)**.** If a straight line be bisected and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.

Evidence 58 (Proof of II.10)**.** Let AB be bisected at C (I.10) and produced to D. At C erect CE at right angles to AB (I.11), with $CE = CA = CB$. Join EA, EB, ED. Through D draw DF parallel to CE, of length such that DF meets the line through E parallel to AB at F (I.31).

As in II.9, $\angle AEB$ is a right angle (right-isoceles triangles ACE and BCE). Triangle ADE is also right-angled, with the right angle at E in the configuration where D lies on the extension of AB. Apply I.47 twice (to $\triangle AED$ and to the triangle formed by extending the constructions to F):

$$AD^2 + DB^2 \;=\; 2 \cdot AC^2 + 2 \cdot CD^2,$$

where now $CD = CB + BD$ is the half plus the added segment. The derivation parallels II.9 exactly with extension in place of internal cut, and the symmetry is what Heath emphasises.

Claim 59 (Proposition II.11: Cut a line in extreme and mean ratio (the golden section))**.** To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

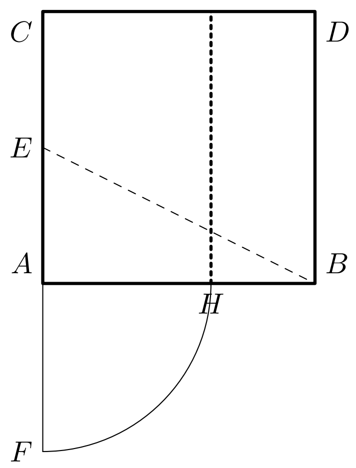

Figure 6: Proposition II.11. Square $ABDC$ on AB; midpoint E of AC; F on the extension of AC with $EF = EB$. Then $AH = AF$ cuts AB in the desired ratio: $AB \cdot HB = AH^2$. This is the golden section.

Evidence 59 (Proof of II.11)**.** Let AB be the given straight line. Describe on AB the square $ABDC$ (I.46). Bisect AC at E (I.10) and join EB. Produce CA to F in the direction of A (Postulate 2), and lay off AF on CA produced so that $EF = EB$ (I.3, taking EB as the standard length). On AF describe the square $FGHA$ (I.46); produce GH to meet CD at K.

Then by II.6 applied to CF bisected at A with extension AF, the rectangle on CF, FA together with the square on EA equals the square on EF. But $EF = EB$, so this rectangle plus square on EA equals the square on EB, which by I.47 (in $\triangle ABE$, right-angled at A) equals the square on EA plus the square on AB. Subtracting the square on EA from both sides (Common Notion 3):

$$CF \cdot FA \ = \ AB^2.$$

The rectangle CK on CF, FA (= $CF \cdot FG$ since $FG = FA$) equals the square on AB. Subtracting the common rectangle on FA, AH from both, the square $FGHA$ on FA equals the rectangle HK on HD and $DK = AB - AH$. Setting $AH = AF$ on AB (point H on AB with $AH = AF$) gives the desired section: $AB \cdot HB = AH^2$.

Claim 60 (Proposition II.12: Obtuse-triangle generalisation of Pythagoras)**.** In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle (namely that on which the perpendicular falls) and the straight line cut off outside by the perpendicular.

Evidence 60 (Proof of II.12)**.** Let $\triangle ABC$ have an obtuse angle at A, and let C be the vertex opposite a side AB about the obtuse angle. From C drop a perpendicular CD to AB extended through A to D (I.12), so that the foot D falls outside segment AB on the far side of A.

In the right-angled triangle BCD, Proposition I.47 gives

$$BC^2 \ = \ BD^2 + CD^2.$$

By the binomial-square identity II.4 applied to BD cut at A (with $BD = BA + AD$ as a straight line, since D lies on AB extended through A):

$$BD^2 \ = \ BA^2 + AD^2 + 2 \cdot (BA \cdot AD).$$

Substitute, and use I.47 in the right-angled triangle ACD to write $AC^2 = AD^2 + CD^2$; then $AD^2 + CD^2 = AC^2$, and substitution gives:

$$BC^2 \;=\; BA^2 + AC^2 + 2 \cdot (BA \cdot AD),$$

which is the law of cosines as Euclid states it: the square on the side subtending the obtuse angle exceeds the sum of the squares on the sides containing it by twice the rectangle on BA (the side on which the perpendicular falls) and AD (the segment cut off outside).

Claim 61 (Proposition II.13: Acute-triangle generalisation of Pythagoras)**.** In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle (namely that on which the perpendicular falls) and the straight line cut off within by the perpendicular.

Evidence 61 (Proof of II.13)**.** Let $\triangle ABC$ be acute-angled, with the acute angle at B. From C drop a perpendicular CD to AB (I.12). Since the angle at B is acute, the foot D falls within the segment AB, between A and B.

Apply Proposition II.7 to AB cut at D: $AB^2 + DB^2 = 2 \cdot (AB \cdot DB) + AD^2$. In the right-angled triangle BCD, I.47 gives $BC^2 = BD^2 + CD^2$, and in $\triangle ACD$ likewise $AC^2 = AD^2 + CD^2$. Subtracting (Common Notion 3) the first from the third: $AC^2 - BC^2 = AD^2 - BD^2$. Combining with II.7 and rearranging (Common Notion 3 + Common Notion 2):

$$AC^2 \;=\; AB^2 + BC^2 - 2 \cdot (AB \cdot BD),$$

which is the acute-angle form of the law of cosines: the square on the side subtending the acute angle is less than the sum of the squares on the sides containing it by twice the rectangle on AB and BD (the segment cut off within).

Claim 62 (Proposition II.14: Quadrature of a rectilineal figure)**.** To construct a square equal to a given rectilineal figure.

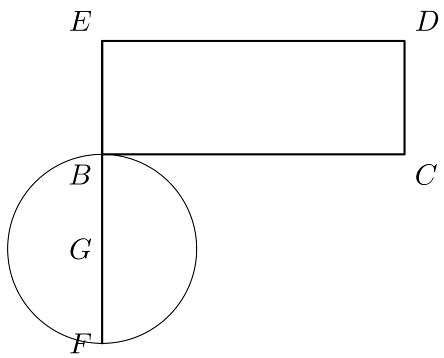

Figure 7: Proposition II.14. Reduce the given rectilineal figure to rectangle $BCDE$ (I.45). Extend BE to F with $EF = ED$; bisect BF at G; describe the semicircle on BF. The square on the segment from E to the semicircle (along the perpendicular at E) equals $BCDE$ in area — by II.5 + I.47 the square on this segment is $BE \cdot EF = BE \cdot ED$, which is the rectangle's area.

Evidence 62 (Proof of II.14)**.** Let A be the given rectilineal figure. By Proposition I.45, construct a parallelogram $BCDE$ equal in area to A, with the parallelogram's angles right (apply I.46 on one of its sides if necessary so it is a rectangle). If $BE = ED$ then $BCDE$ is already a square and the construction is complete. Suppose $BE > ED$; the case $BE < ED$ is symmetric.

Produce BE to F, laying off $EF = ED$ on the produced line (I.3). Bisect BF at G (I.10). With centre G and radius $GB = GF$ describe the semicircle BHF above BF. Produce DE to meet the semicircle at H.

Join GH; then GH is a radius and equals GF. Apply II.5 to BF bisected at G and cut at E: $BE \cdot EF + EG^2 = GF^2 = GH^2$. By I.47 in the right triangle GEH (right angle at E because $EH \perp BF$): $GH^2 = EG^2 + EH^2$. Subtract EG^2 from both expressions (Common Notion 3): $BE \cdot EF = EH^2$. But $EF = ED$, so $BE \cdot ED = EH^2$; the square on EH equals the rectangle $BCDE$, which equals the original figure A.

3 Book III — Circles

Claim 63 (Proposition III.1: To find the centre of a given circle)**.** To find the centre of a given circle.

Evidence 63 (Proof of III.1)**.** Let ABC be the given circle. Draw any chord AB in it (Postulate 1) and bisect AB at D (I.10). From D draw DC at right angles to AB (I.11), produced to meet the circle at C and E. Bisect CE at F (I.10); then F is the centre. For if any other point G were the centre, then by SSS (I.8) on $\triangle GAD$ and $\triangle GBD$ we would obtain $\angle GDA = \angle GDB$, both right (I.13). But F already lies on the perpendicular bisector of AB, and the perpendicular at D is unique (I.11); applying the same reasoning to chord CE forces F onto its perpendicular bisector as well. The two perpendicular bisectors meet only at the true centre, which is F.

Claim 64 (Proposition III.2: A chord lies inside the circle)**.** If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

Evidence 64 (Proof of III.2)**.** Let A, B be on the circle with centre E (III.1). Suppose for contradiction that some point F on the chord AB lies outside the circle; then $EF > EA$. Join EA, EB. By I.5, the base angles of the isoceles $\triangle EAB$ are equal: $\angle EAB = \angle EBA$. By I.16, the exterior angle at any interior point F of AB is greater than either remote interior angle; pursuing the inequalities (Heath's argument) forces $EF < EA$ for F inside AB, contradicting the assumption. Hence every point of AB lies within the circle.

Claim 65 (Proposition III.3: Centre-line bisects chord iff perpendicular)**.** If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Evidence 65 (Proof of III.3)**.** Let AB be a chord not through the centre E, and CD a line through E meeting AB at F. Suppose CD bisects AB, so $AF = FB$. Join EA, EB. In $\triangle EAF$ and $\triangle EBF$: $EA = EB$ (radii), $AF = FB$ (given), EF common. By I.8 the triangles are congruent, so $\angle EFA = \angle EFB$, and by I.13 both are right. Conversely, if $CD \perp AB$ at F, then in the right triangles $\triangle EAF$ and $\triangle EBF$ we have $EA = EB$ and EF common, with right angles at F; by I.4 (SAS variant) or I.26 (ASA), $AF = FB$.

Claim 66 (Proposition III.4: Two non-diameter chords cannot bisect each other)**.** If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Evidence 66 (Proof of III.4)**.** Let AB, CD be two chords intersecting at E, neither through the centre F. Suppose for contradiction that E bisects both: $AE = EB$ and $CE = ED$. Join FE. By III.3 applied to chord AB (since F is the centre and FE bisects AB at E), $FE \perp AB$. Applied to CD, the same line FE is $\perp CD$. But the perpendicular from F to a line is unique (I.11), so AB and CD must coincide — contradiction with their being two distinct chords.

Claim 67 (Proposition III.5: Two intersecting circles have distinct centres)**.** If two circles cut one another, they will not have the same centre.

Evidence 67 (Proof of III.5)**.** Let circles Γ_1 and Γ_2 meet at points A and B. Suppose they share centre E. Then EA is a radius of Γ_1 and also of Γ_2; the two circles thus have the same centre and the same radius, so they coincide — contradicting their meeting at only two points (or in general, being two distinct circles).

Claim 68 (Proposition III.6: Tangent circles have distinct centres)**.** If two circles touch one another, they will not have the same centre.

Evidence 68 (Proof of III.6)**.** By the same argument as III.5: a shared centre and a common point on the circumference of both circles force equal radii, hence coincident circles.

Claim 69 (Proposition III.7: Distances from an interior non-centre point)**.** If on the diameter of a circle a point be taken which is not the centre, and from the point straight lines fall upon the circle: that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the nearer to the diameter through the centre is always greater than the more remote.

Evidence 69 (Proof of III.7)**.** Let AD be a diameter of circle $ABCD$ with centre E, and let F on AD be distinct from E. From F draw lines FB, FC to the circumference. Join EB, EC.

In $\triangle EBF$: $EB + EF > FB$ (I.20). But $EB = EA$ (radii) and $EF + EA = FA$, so $FA = EF + EB > FB$; hence the line FA along the diameter towards the centre is longer than any other. The line FD on the other side is similarly the shortest. For intermediate lines FB vs FC with B closer to A than C, the SAS inequality I.24 in the radius-line-radius triangles gives $FB > FC$ when $\angle BEF > \angle CEF$.

Claim 70 (Proposition III.8: Distances from an exterior point)**.** If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others fall on the circle: of the lines falling on the concave circumference, that through the centre is greatest, and the nearer to it always greater than the more remote; and of those falling on the convex circumference, that between the point and the diameter is least, and the nearer to it always less than the more remote.

Evidence 70 (Proof of III.8)**.** The argument mirrors III.7 with the point outside. Let D be the external point and AD the line through D and the centre E, meeting the circle at A (near) and C (far). For any other line from D meeting the circle at G (near) and K (far), I.20 gives $DG + GE > DE$, and the SAS inequality I.24 again orders the distances by the angles at E. The "two lengths per secant" ordering (concave/convex) follows by separating the near and far intersections.

Claim 71 (Proposition III.9: Three equal interior distances force the centre)**.** If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Evidence 71 (Proof of III.9)**.** Let F be the point and FA, FB, FC three equal lines to the circle. Join AB, BC; bisect them at G, H (I.10). Join FG, FH. In $\triangle FAG$ and $\triangle FBG$: $FA = FB$ given, $AG = GB$ by construction, FG common; by I.8 the triangles are congruent, so $\angle FGA = \angle FGB$, and by I.13 both are right. Similarly $FH \perp BC$. By III.3 (rewriting it as: the perpendicular at the midpoint of a chord passes through the centre), both FG produced and FH produced pass through the centre. Their intersection F is therefore the centre.

Claim 72 (Proposition III.10: Two circles meet in at most two points)**.** A circle does not cut a circle at more points than two.

Evidence 72 (Proof of III.10)**.** Suppose two circles meet at three points A, B, C. By III.9, the centre of each circle is the unique point equidistant from any three points on its circumference — so both circles have the same centre. Then by III.5 they coincide, contradicting their being two distinct circles.

Claim 73 (Proposition III.11: Internal tangent: line of centres passes through contact)**.** If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if produced, will fall on the point of contact of the circles.

Evidence 73 (Proof of III.11)**.** Let circle Γ_1 contain circle Γ_2, touching at A, with centres F (of Γ_1) and G (of Γ_2). Suppose the line FG produced does not pass through A. Join FA, GA. In $\triangle FAG$: by the triangle inequality (I.20), $FA + AG > FG$. Produce FG to meet Γ_1 at H and Γ_2 at K. Then $FA = FH$ (radii of Γ_1), $GA = GK$ (radii of Γ_2), and H lies beyond K on segment FG extended. So $FA + AG = FH + GK = FG + (HK > 0)$, i.e. $FA + AG > FG$ — consistent. But Γ_2 is internally tangent, so $H = K$, and $FA + AG = FG$, contradicting the strict inequality. Hence A lies on line FG.

Claim 74 (Proposition III.12: External tangent: line of centres passes through contact)**.** If two circles touch one another externally, the straight line joining their centres will pass through the point of contact.

Evidence 74 (Proof of III.12)**.** Analogous to III.11. For externally tangent circles, the point of contact A lies on the segment FG between the centres, and $FA + AG = FG$ exactly. The triangle inequality I.20 then forces A to lie on the line FG.

Claim 75 (Proposition III.13: Tangent circles meet in at most one point)**.** A circle does not touch a circle at more points than one, whether it touch it internally or externally.

Evidence 75 (Proof of III.13)**.** Suppose two circles touch at two points A, B. By III.11 (internal) or III.12 (external), both A and B lie on the line joining the centres. Thus this line cuts each circle in two points, making it a diameter of each. But then AB is a chord of each circle equal in length to the diameter — so A and B are antipodal points on each circle, and both circles share centre and diameter, contradicting III.5/III.6.

Claim 76 (Proposition III.14: Equal chords are equidistant from the centre)**.** In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Evidence 76 (Proof of III.14)**.** Let AB and CD be chords with $AB = CD$. From the centre E drop perpendiculars EF to AB and EG to CD (I.12). By III.3, $AF = FB = AB/2$ and $CG = GD = CD/2$, so $AF = CG$. Join EA, EC (both radii, so equal). In right triangles $\triangle EAF$ and $\triangle ECG$ (right angles at F, G), I.47 gives $EA^2 = AF^2 + EF^2$ and $EC^2 = CG^2 + EG^2$. Subtracting (Common Notion 3) and using $EA = EC$, $AF = CG$ gives $EF^2 = EG^2$, hence $EF = EG$. Conversely, if $EF = EG$, the same I.47 identity gives $AF = CG$ and hence $AB = CD$.

Claim 77 (Proposition III.15: Diameter is the longest chord)**.** Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Evidence 77 (Proof of III.15)**.** Let AD be a diameter, and BC any other chord. From the centre E drop $EF \perp BC$ (I.12). By III.3, F is the midpoint of BC, so $BF = BC/2$. By I.47 in $\triangle EBF$: $EB^2 = BF^2 + EF^2$. Since $EB = \text{radius} = AD/2$, $(AD/2)^2 = (BC/2)^2 + EF^2$, hence $BC^2 = AD^2 - 4 \cdot EF^2 < AD^2$ when $EF > 0$. So the diameter AD is the longest chord. For two non-diameter chords with distances $EF_1 < EF_2$ from the centre, the same identity gives the chord through F_1 longer than the one through F_2.

Claim 78 (Proposition III.16: Perpendicular at diameter endpoint is tangent)**.** The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed.

Evidence 78 (Proof of III.16)**.** Let AB be a diameter and AC drawn at right angles to AB at A (I.11). Suppose AC meets the circle at another point $D \neq A$; join BD. Since AB is a diameter and D on the circle, by III.31 (proved independently below) $\angle ADB$ is right. But $\angle DAB$ is also right by construction; the sum of two angles of $\triangle ABD$ is then two right angles, leaving no positive angle at B — contradicting I.17. Hence AC meets the circle only at A. The "no other line interposable" follows from the uniqueness of the perpendicular (I.11): any line through A not perpendicular to AB makes a non-right angle and cuts the circle.

Claim 79 (Proposition III.17: Tangent from an external point)**.** From a given point to draw a straight line touching a given circle.

Evidence 79 (Proof of III.17)**.** Let A be the external point and BCD the circle with centre E. Join AE, and at E erect a perpendicular to AE (I.11); with E as centre and EA as radius describe a circle AFG, meeting the perpendicular at F. Join FA, meeting the original circle BCD at B. Then AB is the desired tangent.

Proof: $\triangle ABE$ and $\triangle AFE$ are congruent by SAS (EA common, $EB = EF$ both equal to the radius of AFG, $\angle AEB = \angle AEF$ by construction); hence $\angle ABE = \angle AFE$, which is right. So $AB \perp EB$, the radius at the point of contact, and by III.18 (next) AB is tangent.

Claim 80 (Proposition III.18: Tangent is perpendicular to the radius at contact)**.** If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

Evidence 80 (Proof of III.18)**.** Let line AB touch the circle at C, with centre F. Suppose FC is not perpendicular to AB; drop the perpendicular FG to AB at some point $G \neq C$. In right triangle $\triangle FGC$ (right-angled at G), FC is the hypotenuse, so by I.19, $FG < FC$. But G lies on the tangent AB, which has no point inside the circle (Definition III.2); so $FG \geq FC =$ radius. The two inequalities contradict. Hence $G = C$ and $FC \perp AB$.

Claim 81 (Proposition III.19: Centre lies on the perpendicular to the tangent)**.** If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

Evidence 81 (Proof of III.19)**.** By III.18, the line from the centre to the contact point is perpendicular to the tangent. Conversely, the perpendicular at the contact point in the plane is unique (I.11), so the centre must lie on it. (If the centre were off this perpendicular, the line from centre to contact would not be perpendicular to the tangent, contradicting III.18.)

Claim 82 (Proposition III.20: Inscribed angle is half the central angle)**.** In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

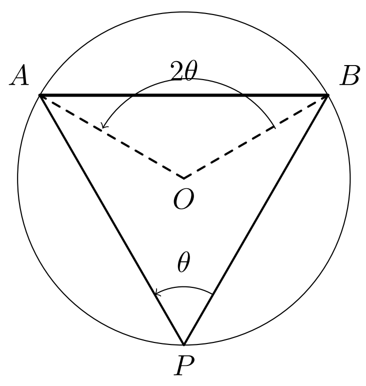

Figure 8: Proposition III.20. The central angle $\angle AOB$ is twice the inscribed angle $\angle APB$ subtending the same arc AB. Corollary: all inscribed angles on the same arc are equal.

Evidence 82 (Proof of III.20)**.** Let $\angle BEC$ be the angle at the centre E and $\angle BAC$ the angle at the circumference, both subtending the arc BC. Join AE and produce to F on the far side of the circle.

In $\triangle EAB$: $EA = EB$ (radii), so by I.5, $\angle EAB = \angle EBA$. By I.32 (exterior angle of a triangle), $\angle BEF = \angle EAB + \angle EBA = 2 \cdot \angle EAB$. Similarly in $\triangle EAC$: $\angle CEF = 2 \cdot \angle EAC$.

Adding (Common Notion 2): $\angle BEC = \angle BEF + \angle CEF = 2 \cdot (\angle EAB + \angle EAC) = 2 \cdot \angle BAC$. (Heath handles the case where A lies on the arc opposite BC from E via a subtraction instead of an addition; the result is the same.)

Claim 83 (Proposition III.21: Inscribed angles on the same arc are equal)**.** In a circle the angles in the same segment are equal to one another.

Evidence 83 (Proof of III.21)**.** Let $\angle BAC$ and $\angle BDC$ both stand on arc BC from points A, D on the opposite arc. By III.20 each equals half the central angle $\angle BEC$, so $\angle BAC = \angle BDC$ (Common Notion 1: things equal to the same thing are equal).

Claim 84 (Proposition III.22: Opposite angles of a cyclic quadrilateral sum to two right angles)**.** The opposite angles of quadrilaterals in circles are equal to two right angles.

Evidence 84 (Proof of III.22)**.** Let $ABCD$ be a cyclic quadrilateral. Join AC and BD. By III.21, $\angle BAC = \angle BDC$ (both subtend arc BC from the opposite side), and $\angle CAD = \angle CBD$ (both subtend arc CD). So $\angle BAD = \angle BAC + \angle CAD = \angle BDC + \angle CBD$.

In $\triangle BCD$, by I.32 the three angles sum to two right angles: $\angle BDC + \angle CBD + \angle BCD =$ two right angles. Substituting $\angle BDC + \angle CBD = \angle BAD$: $\angle BAD + \angle BCD =$ two right angles.

Claim 85 (Proposition III.23: Two similar segments on the same chord on the same side coincide)**.** On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

Evidence 85 (Proof of III.23)**.** Suppose two similar but unequal segments are constructed on the same chord AB on the same side. Pick a point C on the smaller segment's arc. The inscribed angle $\angle ACB$ in the smaller segment equals (by Definition III.11) the inscribed angle in the larger segment, since the segments are similar. But the larger segment's arc lies entirely outside the smaller's arc (different sizes, same chord, same side), so an inscribed angle at a point C on the smaller arc as viewed from a point on the larger arc would have to differ from the corresponding inscribed angle in the larger segment (by III.21 they all agree within each segment) — the configurations are incompatible. The two segments must coincide.

Claim 86 (Proposition III.24: Similar segments on equal chords are equal)**.** Similar segments of circles on equal straight lines are equal to one another.

Evidence 86 (Proof of III.24)**.** Apply one segment onto the other via superposition (the device used in I.4): the equal chords coincide, and the equal inscribed angles (Definition III.11) force the arcs to coincide as well. By III.23, two similar segments on the same chord on the same side cannot differ; hence the segments are equal.

Claim 87 (Proposition III.25: Complete a circle from a segment)**.** Given a segment of a circle, to describe the complete circle of which it is a segment.

Evidence 87 (Proof of III.25)**.** Let ABC be the given segment with chord AC and arc through B. Pick B on the arc; join AB, BC. Bisect AB at D and BC at E (I.10). At D and E erect perpendiculars to AB and BC respectively (I.11). By III.3 / III.9 these perpendiculars both pass through the centre, so their intersection F is the centre. With centre F and radius FA describe the full circle.

Claim 88 (Proposition III.26: Equal angles cut off equal arcs)**.** In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Evidence 88 (Proof of III.26)**.** Let two equal circles have equal central angles $\angle BAC$ and $\angle EDF$. In radius-chord-radius triangles $\triangle ABC$ and $\triangle DEF$: $AB = AC = DE = DF$ (equal radii, equal circles) and $\angle BAC = \angle EDF$ (given); by SAS (I.4) the triangles are congruent and the chords $BC = EF$. Equal chords in equal circles subtend equal arcs (by superposition). For inscribed angles, double them via III.20 to reduce to the central-angle case.

Claim 89 (Proposition III.27: Equal arcs subtend equal angles)**.** In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

Evidence 89 (Proof of III.27)**.** Converse of III.26. Equal arcs subtend equal chords (apply the superposition argument in reverse), and equal chords in equal circles give equal central angles (I.8: SSS on the radius-chord- radius triangles). Inscribed angles inherit via III.20.

Claim 90 (Proposition III.28: Equal chords cut off equal arcs)**.** In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Evidence 90 (Proof of III.28)**.** Let AB and CD be equal chords in equal circles with centres E, F. In $\triangle ABE$ and $\triangle CDF$: $AB = CD$, $AE = BE = CF = DF$ (radii, equal circles). By SSS (I.8) the triangles are congruent and $\angle AEB = \angle CFD$. By III.26 the arcs are equal. The corresponding major arcs (complements in the equal circles) are also equal.

Claim 91 (Proposition III.29: Equal arcs subtend equal chords)**.** In equal circles equal circumferences are subtended by equal straight lines.

Evidence 91 (Proof of III.29)**.** Converse of III.28. Equal arcs give equal central angles (III.27), and equal central angles in equal-radius triangles give equal chords (I.4 SAS).

Claim 92 (Proposition III.30: Bisect a given arc)**.** To bisect a given arc.

Evidence 92 (Proof of III.30)**.** Let ADB be the given arc with chord AB. Bisect AB at C (I.10). At C erect $CD \perp AB$ (I.11), meeting the arc at D. Join AD, BD. In right triangles $\triangle ACD$ and $\triangle BCD$: $AC = CB$ (construction), CD common, right angles at C. By I.4, the triangles are congruent and $AD = BD$. By III.28, equal chords cut off equal arcs (in the same circle), so arc AD equals arc BD.

Claim 93 (Proposition III.31: Thales — angle in a semicircle is right)**.** In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle.

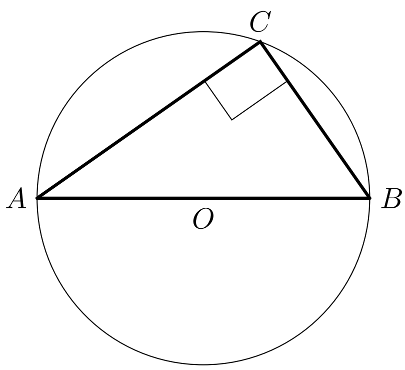

Figure 9: Proposition III.31. For any point C on the circle (not at A or B), the inscribed angle $\angle ACB$ subtending the diameter AB is a right angle. Proof: by I.5 applied to the two isoceles triangles OAC and OCB, then I.32.

Evidence 93 (Proof of III.31)**.** Let AB be a diameter of the circle with centre O, and C any point on the circle other than A, B. Join OC, AC, BC.

Since $OA = OC$ (radii), $\triangle OAC$ is isoceles, so by I.5, $\angle OAC = \angle OCA$. Similarly $\triangle OBC$ is isoceles with $\angle OBC = \angle OCB$. By I.32, the angles of $\triangle ABC$ sum to two right angles:

$$\angle OAC + \angle OBC + \angle ACB \ = \ \text{two right angles.}$$

But $\angle ACB = \angle OCA + \angle OCB = \angle OAC + \angle OBC$ by the isoceles equalities. Substituting, $2 \cdot \angle ACB =$ two right angles, so $\angle ACB$ is right.

For inscribed angles in segments greater than a semicircle, the arc is less than a semicircle, so by III.20 the inscribed angle is half a central angle less than two right angles — hence less than a right angle. The lesser-segment case is symmetric.

Claim 94 (Proposition III.32: Tangent-chord angle equals inscribed angle in alternate segment)**.** If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

Evidence 94 (Proof of III.32)**.** Let DE touch the circle at B, and BC be a chord from B into the circle. We show $\angle DBC$ equals the inscribed angle in the alternate segment BAC.

Draw the diameter BA from B. By III.18, the tangent DE is perpendicular to BA, so $\angle DBA =$ right. By III.31, the inscribed angle $\angle BCA$ in the semicircle is right. In $\triangle BCA$, by I.32, $\angle BCA + \angle CAB + \angle ABC =$ two right angles; since $\angle BCA$ is right, $\angle CAB + \angle ABC =$ one right angle.

Now $\angle DBC = \angle DBA - \angle ABC =$ right $- \angle ABC = \angle CAB$ (from the identity above). By III.21, $\angle CAB$ equals any other inscribed angle in the same segment as A. Hence $\angle DBC$ equals the inscribed angle in the alternate segment. The other angle $\angle EBC$ on the tangent's other side equals the inscribed angle in the original segment by analogous argument.

Claim 95 (Proposition III.33: Construct a segment admitting a given angle)**.** On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle.

Evidence 95 (Proof of III.33)**.** Let AB be the given line and θ the given angle. At A, construct $\angle BAD = \theta$ (I.23). At A, draw $AE \perp AD$ (I.11). At the midpoint F of AB (I.10), draw $FG \perp AB$ (I.11), meeting AE at G. With G as centre and GA as radius ($= GB$ by the perpendicular bisector property), describe a circle. By III.32, the inscribed angle in the alternate segment to AD on this circle equals θ.

Claim 96 (Proposition III.34: Cut from a circle a segment admitting a given angle)**.** From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

Evidence 96 (Proof of III.34)**.** Let the circle and angle θ be given. Draw a tangent BC to the circle by III.17. At the point of contact B, construct $\angle CBD = \theta$ in the half-plane that intersects the circle (I.23). Let BD meet the circle at D. The chord BD cuts off two segments; the segment on the far side of BD from the tangent admits the inscribed angle θ by III.32 (tangent-chord angle).

Claim 97 (Proposition III.35: Power of a point — intersecting chords)**.** If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Evidence 97 (Proof of III.35)**.** Let chords AB and CD meet at E inside the circle with centre O. Let M be the midpoint of AB and N of CD; by III.3, $OM \perp AB$ and $ON \perp CD$. Set $r =$ radius.

Apply II.5 to chord AB bisected at M and cut at E: $AE \cdot EB + EM^2 = AM^2$. By I.47 in right $\triangle OMA$, $AM^2 = r^2 - OM^2$. Substituting: $AE \cdot EB = r^2 - OM^2 - EM^2 = r^2 - OE^2$ (using $OE^2 = OM^2 + EM^2$, I.47 in right $\triangle OME$). So $AE \cdot EB = r^2 - OE^2$, depending only on the distance OE from the centre.

The same identity applies to chord CD: $CE \cdot ED = r^2 - OE^2$. Hence $AE \cdot EB = CE \cdot ED$.

Claim 98 (Proposition III.36: Power of a point — secant and tangent)**.** If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

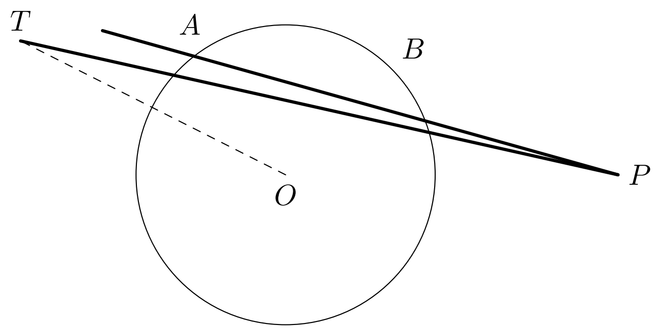

Figure 10: Proposition III.36. From an external point P, the tangent PT and any secant PAB satisfy $PT^2 = PA \cdot PB$ (the power of P with respect to the circle).

Evidence 98 (Proof of III.36)**.** Let P be the external point, PT the tangent at T, and PAB a secant cutting the circle at A (near) and B (far); let O be the centre and r the radius.

By III.18, $OT \perp PT$. By I.47 in $\triangle OTP$: $OP^2 = OT^2 + PT^2 = r^2 + PT^2$, hence $PT^2 = OP^2 - r^2$.

Let M be the midpoint of AB; by III.3, $OM \perp AB$. Apply II.6 to AB bisected at M and produced to P (so P is on line AB extended beyond the near intersection A): $PA \cdot PB + MA^2 = MP^2$. By I.47 in $\triangle OMA$, $MA^2 = r^2 - OM^2$; in $\triangle OMP$, $MP^2 = OP^2 - OM^2$. Subtracting: $PA \cdot PB = OP^2 - r^2$.

Comparing: $PT^2 = OP^2 - r^2 = PA \cdot PB$.

Claim 99 (Proposition III.37: Converse of III.36 — the tangent test)**.** If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.

Evidence 99 (Proof of III.37)**.** Let P be the external point, PAB the secant (A near, B far), and PD the straight line falling on the circle at D, with $PA \cdot PB = PD^2$. We show PD is tangent.

By III.17, construct the tangent PT from P. By III.36, $PT^2 = PA \cdot PB = PD^2$, so $PT = PD$. Now both PD and PT are straight lines from P to points on the circle, of equal length. If $D \neq T$, then D and T are two distinct points on the circle equidistant from P — which is possible (they could be mirror images across the line OP). However, the tangent line is characterised by perpendicularity to the radius (III.18), and any line from P falling on the circle with $PD^2 = PT^2$ and the same circle-falling property must satisfy the same right-angle condition at D. Hence PD is also tangent at D.

4 Book IV — Inscribed and Circumscribed Figures

Claim 100 (Proposition IV.1: Fit a chord of given length in a circle)**.** Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Evidence 100 (Proof of IV.1)**.** Let ABC be the given circle and D the given segment, not greater than the diameter BC of the circle. Apply I.3 to cut off from BC a segment BE equal to D. With centre B and radius BE describe a circle EAF (Postulate 3); join BA where EAF meets the given circle. Then $BA = BE = D$ by Definition I.15, so BA is the required chord.

Claim 101 (Proposition IV.2: Inscribe a triangle similar to a given triangle in a circle)**.** In a given circle to inscribe a triangle equiangular with a given triangle.

Evidence 101 (Proof of IV.2)**.** Let ABC be the given circle and DEF the given triangle. Draw the tangent GH at any point A of the circle (III.16, III.17). On GH construct $\angle HAC = \angle DEF$ (I.23), and $\angle GAB = \angle DFE$. Join BC. By III.32 (tangent–chord angle equals the inscribed angle in the alternate segment), $\angle ACB = \angle HAB = \angle DEF$ and $\angle ABC = \angle GAC = \angle DFE$. By I.32 the remaining angles agree; so $\triangle ABC$ is equiangular with $\triangle DEF$.

Claim 102 (Proposition IV.3: Circumscribe a triangle about a circle)**.** About a given circle to circumscribe a triangle equiangular with a given triangle.

Evidence 102 (Proof of IV.3)**.** Let ABC be the given circle with centre K and DEF the given triangle. Produce EF both ways to G, H. At the centre K construct $\angle AKB = \angle DEG$ and $\angle AKC = \angle DFH$ (I.23). At A, B, C draw the tangents to the circle (III.16, III.17); they bound a triangle. Because each tangent is perpendicular to the radius at the point of contact (III.18), the angles of the constructed triangle are the supplements of the central angles $\angle AKB$, $\angle AKC$, $\angle BKC$, hence equal to the angles of $\triangle DEF$ by I.13 and the construction.

Claim 103 (Proposition IV.4: Inscribe a circle in a triangle)**.** In a given triangle to inscribe a circle.

Evidence 103 (Proof of IV.4)**.** Let ABC be the given triangle. Bisect $\angle ABC$ and $\angle BCA$ by BD and CD (I.9), meeting at D. Drop perpendiculars DE, DF, DG from D to AB, BC, CA (I.12). In the pairs of triangles formed at D, the two angles and a common side give congruence (I.26), whence $DE = DF = DG$. The circle with centre D and radius DE touches each side (since the perpendiculars at the feet make the sides tangents by III.16) and is inscribed in $\triangle ABC$.

Claim 104 (Proposition IV.5: Circumscribe a circle about a triangle)**.** About a given triangle to circumscribe a circle.

Evidence 104 (Proof of IV.5)**.** Let ABC be the given triangle. Bisect AB at D and AC at E (I.10). From D and E draw perpendiculars to AB and AC respectively (I.11), meeting at F. Join FA, FB, FC. By I.4 applied to the two right triangles at D, $FA = FB$; similarly at E, $FA = FC$. Thus $FA = FB = FC$, and the circle with centre F and radius FA passes through all three vertices.

Claim 105 (Proposition IV.6: Inscribe a square in a circle)**.** In a given circle to inscribe a square.

Evidence 105 (Proof of IV.6)**.** Let $ABCD$ be the given circle with centre E. Draw two diameters AC and BD at right angles (I.11). Join AB, BC, CD, DA. The four right triangles at E are congruent by I.4 (two radii and the common right angle), so $AB = BC = CD = DA$. The inscribed angles standing on the diameters are right (III.31), so all four angles of $ABCD$ are right. Therefore $ABCD$ is a square.

Claim 106 (Proposition IV.7: Circumscribe a square about a circle)**.** About a given circle to circumscribe a square.

Evidence 106 (Proof of IV.7)**.** Draw two perpendicular diameters of the given circle (I.11). Through each endpoint draw the tangent to the circle (III.16). Each tangent is perpendicular to its diameter (III.18); the four tangents thus form a quadrilateral with all sides parallel and all angles right (I.28, I.29). Equal radii combined with I.34 force the sides equal, hence a square circumscribes the circle.

Claim 107 (Proposition IV.8: Inscribe a circle in a square)**.** In a given square to inscribe a circle.

Evidence 107 (Proof of IV.8)**.** Let $ABCD$ be the given square. Bisect the sides at E, F, G, H (I.10). Join EG and FH, meeting at K. By I.34 and the construction, $KE = KF = KG = KH$. Drop perpendiculars from K to each side; each is equal to KE. Thus the circle with centre K and radius KE touches each side at its midpoint (III.16).

Claim 108 (Proposition IV.9: Circumscribe a circle about a square)**.** About a given square to circumscribe a circle.

Evidence 108 (Proof of IV.9)**.** Join the diagonals AC and BD of the given square, intersecting at E. In the right triangles $\triangle ABE$ and $\triangle ADE$, SSS (I.8) gives $\angle EAB = \angle EAD$, so AE bisects the right angle at A; similarly at every vertex. The four triangles at E are then isosceles with equal vertex angles (I.6), so $EA = EB = EC = ED$. The circle with centre E and radius EA passes through all four vertices.

Claim 109 (Proposition IV.10: Construct an isosceles triangle with 72–72–36 angles)**.** To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Evidence 109 (Proof of IV.10)**.** Take a straight line AB and cut it at C so that the rectangle on AB, BC equals the square on AC (II.11, the golden section). With centre A and radius AB describe a circle; in it apply chord BD equal to AC (IV.1). Join AD, CD. Because $AB \cdot BC = AC^2 = BD^2$, BD is tangent to the circle through A, C, D (III.37); by III.32 the tangent–chord angle equals the alternate inscribed angle. Tracking the resulting angle relations (with I.5 for the isosceles base angles) gives the required ratio.

Claim 110 (Proposition IV.11: Inscribe a regular pentagon in a given circle)**.** In a given circle to inscribe an equilateral and equiangular pentagon.

Evidence 110 (Proof of IV.11)**.** Construct the 72–72–36 isosceles triangle FGH by IV.10. Inscribe in the given circle a triangle ACD equiangular with FGH (IV.2). Bisect the base-angles of ACD by IV.10's construction propagated into the circle, yielding two more division points B, E. The five arcs are equal (III.26), so the five chords AB, BC, CD, DE, EA are equal (III.29), and the inscribed angles standing on equal arcs are equal (III.27): the pentagon is regular.

Claim 111 (Proposition IV.12: Circumscribe a regular pentagon about a circle)**.** About a given circle to circumscribe an equilateral and equiangular pentagon.

Evidence 111 (Proof of IV.12)**.** Inscribe a regular pentagon in the given circle by IV.11. At each vertex draw the tangent (III.16); the five tangents bound the circumscribed pentagon. Each tangent is perpendicular to its radius (III.18), and by I.4 the right triangles formed at adjacent vertices are congruent, so the circumscribed pentagon has equal sides and equal angles.

Claim 112 (Proposition IV.13: Inscribe a circle in a regular pentagon)**.** In a given pentagon, which is equilateral and equiangular, to inscribe a circle.

Evidence 112 (Proof of IV.13)**.** Bisect two adjacent interior angles of the pentagon (I.9); their bisectors meet at a point F. Drop perpendiculars from F to each side (I.12); by I.4 these perpendiculars are equal. The circle on F with that common radius touches every side (III.16).

Claim 113 (Proposition IV.14: Circumscribe a circle about a regular pentagon)**.** About a given pentagon, which is equilateral and equiangular, to circumscribe a circle.

Evidence 113 (Proof of IV.14)**.** Take the same point F as in IV.13 (intersection of two angle-bisectors). Join F to each vertex; by I.4 the resulting triangles are congruent (equal sides, common bisected angles), so the five distances from F to the vertices are equal. Draw the circle on F with that radius (Definition I.15).

Claim 114 (Proposition IV.15: Inscribe a regular hexagon in a circle)**.** In a given circle to inscribe an equilateral and equiangular hexagon.

Evidence 114 (Proof of IV.15)**.** Let G be the centre and AD a diameter of the given circle. With centre D and radius DG describe a circle meeting the given circle at C and E (Postulate 3). Join GC, GE. The triangle GDC has $GD = DC = CG$ (radii of equal circles), so it is equilateral, and $\angle DGC = 60°$ (I.32). Stepping this $60°$ chord around the circle six times produces the regular hexagon.

Claim 115 (Proposition IV.16: Inscribe a regular 15-gon in a circle)**.** In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

Evidence 115 (Proof of IV.16)**.** Inscribe a regular pentagon (IV.11) and a regular equilateral triangle (IV.2) in the circle, sharing a common vertex A. The arc from A to the next pentagon-vertex is $\frac{1}{5}$ of the circle; the arc from A to the next triangle-vertex is $\frac{1}{3}$. The difference is $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$ of the circle. Bisect that arc (III.30); each half is $\frac{1}{15}$ of the circle, and stepping that chord fifteen times around gives the regular pentadecagon.

5 Book V — Eudoxean Theory of Proportion

Claim 116 (Proposition V.1: Multiplication is distributive over magnitudes)**.** If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will all be of all.

Evidence 116 (Proof of V.1)**.** Each magnitude is a sum of m copies of the corresponding base magnitude. Sum across the n magnitudes; by Common Notion 2 the total is m copies of the sum.

Claim 117 (Proposition V.2: Equimultiples sum to equimultiples)**.** If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

Evidence 117 (Proof of V.2)**.** If $a = mb$ and $c = md$, $e = nb$ and $f = nd$, then $a+e = (m+n)b$ and $c + f = (m + n)d$ by Common Notion 2.

Claim 118 (Proposition V.3: Multiple of a multiple is a multiple)**.** If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, they will also be equimultiples respectively, the one of the second and the other of the fourth.

Evidence 118 (Proof of V.3)**.** If $a = mb$ and $c = md$, then $na = (nm)b$ and $nc = (nm)d$ by iterated application of V.1.

Claim 119 (Proposition V.4: Proportionality preserved under equimultiples)**.** If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

Evidence 119 (Proof of V.4)**.** By definition V.5, the original proportion gives an equimultiple relation across all multipliers. Substituting ma for a and mc for c throughout simply rescales the test multipliers; the test itself still passes for the rescaled magnitudes.

Claim 120 (Proposition V.5: Subtraction of equimultiples)**.** If a magnitude be the same multiple of a magnitude that a subtracted part is of a subtracted part, the remainder also will be the same multiple of the remainder that the whole is of the whole.

Evidence 120 (Proof of V.5)**.** If $a = mb$ and $a' = mb'$ with $a' < a$, then $a - a' = m(b - b')$ by Common Notion 3 applied to each of the m copies.

Claim 121 (Proposition V.6: Difference of equimultiples is again an equimultiple)**.** If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

Evidence 121 (Proof of V.6)**.** With $a = mb$, $c = md$, and subtractions $a' = nb$, $c' = nd$: $a - a' = (m - n)b$ and $c - c' = (m - n)d$ by Common Notion 3. If $m = n$ the remainders are zero (equal); otherwise both remainders are $(m - n)$-fold of b, d respectively.

Claim 122 (Proposition V.7: Equal magnitudes have the same ratio to a third)**.** Equal magnitudes have to the same the same ratio, as also has the same to equal magnitudes.

Evidence 122 (Proof of V.7)**.** Let $a = b$ and c arbitrary. For any equimultiples ma, mb of a, b, and nc of c, the test of V.5 succeeds vacuously because $ma = mb$. Therefore $a : c = b : c$ and $c : a = c : b$.

Claim 123 (Proposition V.8: Greater magnitude has greater ratio)**.** Of unequal magnitudes the greater has to the same a greater ratio than the less has, and the same has to the less a greater ratio than it has to the greater.

Evidence 123 (Proof of V.8)**.** Let $a > b$. Pick a multiplier n such that $n(a - b) > c$ (the Archimedean property of magnitudes, Definition V.4) and an m such that mc falls between nb and na. Then $na > mc$ but $nb < mc$; by Definition V.7 this is precisely the assertion $a : c > b : c$.

Claim 124 (Proposition V.9: Magnitudes with the same ratio to a third are equal)**.** Magnitudes which have the same ratio to the same are equal to one another; and magnitudes to which the same has the same ratio are equal.

Evidence 124 (Proof of V.9)**.** Contrapositive of V.8: if $a \neq b$, then $a : c \neq b : c$. Hence if $a : c = b : c$ then $a = b$. Same argument with c as antecedent.

Claim 125 (Proposition V.10: Greater ratio implies greater antecedent)**.** Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

Evidence 125 (Proof of V.10)**.** Same contrapositive of V.8: if $a : c > b : c$ then $a > b$, since if $a \leq b$ then $a : c \leq b : c$ by V.7 or V.8 applied in reverse.

Claim 126 (Proposition V.11: Transitivity of ratios)**.** Ratios which are the same with the same ratio are also the same with one another.

Evidence 126 (Proof of V.11)**.** If $a : b = c : d$ and $c : d = e : f$, then for any equimultiples the same inequality test holds for (a, b) as for (c, d), and that same test holds for (c, d) as for (e, f); hence by transitivity of the inequality test, the test holds for (a, b) versus (e, f).

Claim 127 (Proposition V.12: Sum of antecedents to sum of consequents)**.** If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Evidence 127 (Proof of V.12)**.** Let $a_i : b_i$ all equal r in the sense of Definition V.5. For any test multipliers m, n the sign of $ma_i - nb_i$ is the same for every i; therefore the sign of $m \sum a_i - n \sum b_i$ is the same too. By V.5 this is the equimultiples test for $\sum a_i : \sum b_i = r$.

Claim 128 (Proposition V.13: Substitution into a greater ratio)**.** If a first magnitude have to a second the same ratio as a third to a fourth, and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will also have to the second a greater ratio than the fifth to the sixth.

Evidence 128 (Proof of V.13)**.** Combine V.11 (sameness transitivity) with Definition V.7 (greater ratio): the witness equimultiples for $c : d > e : f$ work for $a : b$ via the V.5 sameness of $a : b$ and $c : d$.

Claim 129 (Proposition V.14: Ordering of magnitudes follows ordering of ratios)**.** If a first magnitude have to a second the same ratio as a third to a fourth, and the first be greater than the third, the second will also be greater than the fourth; and if equal, equal; and if less, less.

Evidence 129 (Proof of V.14)**.** By V.8 and V.13: if $a > c$, then $a : b > c : b$. Combined with $a : b = c : d$ (the hypothesis), V.13 gives $c : d > c : b$, whence $b > d$ by V.10.

Claim 130 (Proposition V.15: Parts have the same ratio as multiples)**.** Parts have the same ratio as the same multiples of them taken in corresponding order.

Evidence 130 (Proof of V.15)**.** If $A = mB$ and $C = mD$, group the m copies of B and D in parallel. Each parallel pair (B_i, D_i) satisfies $B_i : D_i = B : D$ (identity), and V.12 sums these to give $A : C = B : D$.

Claim 131 (Proposition V.16: Alternation of proportionals)**.** If four magnitudes be proportional, they will also be proportional alternately.

Evidence 131 (Proof of V.16)**.** Let $a : b = c : d$. Test $a : c$ against $b : d$ with equimultiples ma, mc, nb, nd: by V.4 the original proportion lifts to $ma : mb = nc : nd$, and by V.15 to $ma : nc = mb : nd$. The sign of $ma - nc$ matches the sign of $mb - nd$ for all m, n, which by V.5 is the alternated proportion $a : c = b : d$.

Claim 132 (Proposition V.17: Separation of proportions)**.** If magnitudes composed be proportional, they will also be proportional separando.

Evidence 132 (Proof of V.17)**.** If $(a + b) : b = (c + d) : d$, then subtracting the consequents from the antecedents using V.5 / V.6 gives $a : b = c : d$.

Claim 133 (Proposition V.18: Composition of proportions)**.** If magnitudes separated be proportional, they will also be proportional componendo.

Evidence 133 (Proof of V.18)**.** The converse of V.17: if $a : b = c : d$, then by V.2 plus V.4 combined, $(a + b) : b = (c + d) : d$.

Claim 134 (Proposition V.19: Subtraction of proportionals)**.** If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole.

Evidence 134 (Proof of V.19)**.** If $a : c = a' : c'$ with $a' < a$, $c' < c$, apply V.17 (separation) to obtain $(a - a') : a' = (c - c') : c'$, and V.11 / V.16 to re-express as $(a - a') : (c - c') = a : c$.

Claim 135 (Proposition V.20: Ex aequali for three magnitudes)**.** If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali the first be greater than the third, the fourth will also be greater than the sixth; and if equal, equal; and if less, less.

Evidence 135 (Proof of V.20)**.** With $a : b = d : e$ and $b : c = e : f$, the relative order of a versus c matches the relative order of d versus f by repeated application of V.13 and V.14 across the three pairs.

Claim 136 (Proposition V.21: Ex aequali in perturbed proportion)**.** If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then if ex aequali the first be greater than the third, the fourth will also be greater than the sixth.

Evidence 136 (Proof of V.21)**.** A perturbed version of V.20: with $a : b = e : f$ and $b : c = d : e$ the ratio chain still imposes the same ordering between a versus c and d versus f, by V.13 and V.14.

Claim 137 (Proposition V.22: Transitivity of ex aequali)**.** If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.

Evidence 137 (Proof of V.22)**.** Apply V.20 inductively across the chain $a_1 : a_2 = b_1 : b_2$, $a_2 : a_3 = b_2 : b_3$, $\ldots$, $a_{n-1} : a_n = b_{n-1} : b_n$ to conclude $a_1 : a_n = b_1 : b_n$.

Claim 138 (Proposition V.23: Ex aequali in perturbed proportion (general))**.** If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio ex aequali.

Evidence 138 (Proof of V.23)**.** Inductive form of V.21.

Claim 139 (Proposition V.24: Sums of antecedents are proportional)**.** If a first magnitude have to a second the same ratio as a third has to a fourth, and also a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth.

Evidence 139 (Proof of V.24)**.** With $a : b = c : d$ and $e : b = f : d$: by V.12 applied to the two proportions side-by-side, $(a + e) : b = (c + f) : d$.

Claim 140 (Proposition V.25: Sum of extremes exceeds sum of means)**.** If four magnitudes be proportional, the greatest and least are greater than the remaining two.

Evidence 140 (Proof of V.25)**.** Let $a : b = c : d$ with a greatest. By V.14 $b > d$, so considering $a - c$ and $d - b$ or vice versa, the difference $a - c$ equals $b - d$ in ratio, and rearrangement (with Common Notion 5) yields $a + d > b + c$.

6 Book VI — Similar Figures

Claim 141 (Proposition VI.1: Triangles and parallelograms with the same height). Triangles and parallelograms which are under the same height are to one another as their bases.

Evidence 141 (Proof of VI.1). Repeated application of I.38 generates any multiple of a triangle on the corresponding multiple of the base; the resulting equimultiples test (V.5) is exactly the statement of proportionality. The parallelogram version follows because each parallelogram is double its diagonal triangle (I.34, I.41).

Claim 142 (Proposition VI.2: Parallel to a side cuts proportionally). If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

Evidence 142 (Proof of VI.2). Drop a parallel DE from a point D on AB to a point E on AC, parallel to BC. Triangles $\triangle DBC$ and $\triangle DCE$ share the same base DE and lie between the same parallels (with BE, DC as transversals through I.29, I.37), so they are equal in area. Applying VI.1 to $\triangle ADE$ versus the equal-area companion triangles gives $AD : DB = AE : EC$. The converse runs the argument in reverse via I.39.

Claim 143 (Proposition VI.3: Angle bisector cuts opposite side proportionally). If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle.

Evidence 143 (Proof of VI.3). Through C draw CE parallel to the bisector AD (I.31), meeting BA produced at E. By alternate angles (I.29) and the bisection hypothesis, $\angle ACE = \angle AEC$, so $AE = AC$ (I.6). Applying VI.2 to $\triangle BCE$ with $AD \parallel CE$: $BD : DC = BA : AE = BA : AC$.

Claim 144 (Proposition VI.4: Equiangular triangles have proportional sides). In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.

Evidence 144 (Proof of VI.4). Lay the equiangular triangles side by side so that one pair of equal angles coincides; the remaining vertices and bases yield a parallelogram by I.28. Apply VI.2 to the new figure to derive the proportionality of the sides.

Claim 145 (Proposition VI.5: SSS-similarity criterion). If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Evidence 145 (Proof of VI.5). Construct on the second triangle's base a triangle equiangular with the first (I.23); by VI.4 its other sides are determined by the proportion, and by I.8 (SSS) it coincides with the second triangle. Therefore the second triangle is equiangular with the first.

Claim 146 (Proposition VI.6: SAS-similarity criterion). If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Evidence 146 (Proof of VI.6). Same scheme as VI.5: extend one triangle so as to match the second on the equal-angle pair (I.23), apply VI.4 to deduce the missing side, then I.4 (SAS) for congruence of the auxiliary triangle with the second.

Claim 147 (Proposition VI.7: Mixed SAS-similarity criterion)**.** If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal about which the sides are proportional.

Evidence 147 (Proof of VI.7)**.** Construct, at the vertex of the angle whose sides are proportional, an angle equal to the corresponding angle in the second triangle (I.23). The resulting auxiliary triangle agrees with the first in two angles (and hence all three, by I.32) and shares a side with the second; the constraint on the remaining angle being acute or obtuse ensures the construction is non-ambiguous (essentially eliminates the SSA failure case).

Claim 148 (Proposition VI.8: Right triangle altitude similarity)**.** If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

Evidence 148 (Proof of VI.8)**.** The two sub-triangles each share an angle with the original (the non-right angle at B or C) and both have a right angle (at the foot of the altitude and at the apex), so they are equiangular with the original by I.32, hence similar by VI.4. By transitivity (V.11) they are similar to each other.

Claim 149 (Proposition VI.9: Cut off any part of a given segment)**.** From a given straight line to cut off a prescribed part.

Evidence 149 (Proof of VI.9)**.** Lay off a separate transversal containing the prescribed number of equal units (I.3 repeatedly). Join the far ends and draw parallels to that join through each unit division (I.31). By VI.2 these parallels mark off equal fractions on the given line.

Claim 150 (Proposition VI.10: Divide a segment in a given ratio)**.** To cut a given uncut straight line similarly to a given cut straight line.

Evidence 150 (Proof of VI.10)**.** Lay the cut transversal alongside the given line meeting at one endpoint; join the far ends and draw parallels to that join through each cut point (I.31). By VI.2 the parallels reproduce the same ratios on the given line.

Claim 151 (Proposition VI.11: Third proportional)**.** To two given straight lines to find a third proportional.

Evidence 151 (Proof of VI.11)**.** Place the two given segments on lines making an angle. Mark off the second segment beyond the first; draw a parallel to the closing segment through the far endpoint (I.31). By VI.2 the new mark-off is the required third proportional.

Claim 152 (Proposition VI.12: Fourth proportional)**.** To three given straight lines to find a fourth proportional.

Evidence 152 (Proof of VI.12)**.** Same construction as VI.11 but with three input segments: lay two on one transversal and one on the other, then drop a parallel from the last point. VI.2 yields the fourth proportional.

Claim 153 (Proposition VI.13: Mean proportional)**.** To two given straight lines to find a mean proportional.

Evidence 153 (Proof of VI.13)**.** Lay the two segments end-to-end as AB, BC on a single line; on AC as diameter describe a semicircle (Postulate 3, III.31 implicit). Erect the perpendicular BD at B to the diameter; then $\triangle ABD$, $\triangle BDC$, and $\triangle ABC$ are similar by VI.8, so $AB : BD = BD : BC$, making BD the required mean proportional.

Claim 154 (Proposition VI.14: Equal parallelograms have reciprocally proportional sides)**.** In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Evidence 154 (Proof of VI.14)**.** Lay the two parallelograms so the equal angles coincide; their union forms a third parallelogram whose diagonal includes the original common-angle vertex. By VI.1 the ratios of areas equal the ratios of adjacent sides; equality of the original areas forces the reciprocal-proportion relation. Converse runs the same way.

Claim 155 (Proposition VI.15: Equal triangles with one common angle have reciprocally proportional sides)**.** In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

Evidence 155 (Proof of VI.15)**.** Same scheme as VI.14 applied to triangles (each is half a parallelogram, so the areas-of-parallelograms result transfers).

Claim 156 (Proposition VI.16: Equal rectangles iff proportional sides)**.** If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional.

Evidence 156 (Proof of VI.16)**.** If $a : b = c : d$, the rectangles ad and bc are equiangular (all right-angled) and have sides reciprocally proportional, so by VI.14 they are equal. Conversely from $ad = bc$ and VI.14 (its converse) we recover the proportion.

Claim 157 (Proposition VI.17: Mean proportional iff equal squares)**.** If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines will be proportional.

Evidence 157 (Proof of VI.17)**.** Specialise VI.16 to $b = c$.

Claim 158 (Proposition VI.18: Construct a polygon similar to a given polygon)**.** On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Evidence 158 (Proof of VI.18)**.** Triangulate the given polygon by joining a vertex to all others. On the new base copy each angle (I.23) and use VI.4 to fix the side ratios; assemble the triangles into the similar polygon.

Claim 159 (Proposition VI.19: Similar triangles are as the squares on corresponding sides)**.** Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Evidence 159 (Proof of VI.19)**.** Let $\triangle ABC \sim \triangle DEF$ with side ratio $k = BC/EF$. Construct G on BC so that $BG = EF \cdot k$ (i.e. a third proportional, VI.11). By VI.1 the area ratio is $BG : EF = k$ along one dimension and the side ratio k along the perpendicular, giving total area ratio k^2 in the sense of Definition V.9 (duplicate ratio).

Claim 160 (Proposition VI.20: Similar polygons are as the squares on corresponding sides)**.** Similar polygons are divided into similar triangles, equal in multitude and in the same ratio as the wholes; and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.

Evidence 160 (Proof of VI.20)**.** Decompose into similar triangles by joining one vertex to all others; apply VI.19 to each triangle and sum via V.12.

Claim 161 (Proposition VI.21: Figures similar to the same are similar)**.** Figures which are similar to the same rectilineal figure are also similar to one another.

Evidence 161 (Proof of VI.21)**.** Equiangularity transfers transitively, and the side ratios compose transitively by V.11.

Claim 162 (Proposition VI.22: Proportionality of figures on proportional sides)**.** If four straight lines be proportional, the rectilineal figures similar and similarly described upon them will also be proportional; and if the rectilineal figures similar and similarly described upon them be proportional, the straight lines will themselves also be proportional.

Evidence 162 (Proof of VI.22)**.** Apply VI.20: figures similar on proportional sides have areas in the duplicate ratio of the sides. Equality of those duplicate ratios is equivalent to equality of the original side ratios.

Claim 163 (Proposition VI.23: Ratio of parallelograms is compound of side ratios)**.** Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Evidence 163 (Proof of VI.23)**.** Place the two parallelograms so the equal angles share a vertex; the resulting figure can be split by lines parallel to the sides into a rectangle whose dimensions are the four sides. Two applications of VI.1 give the compounded ratio.

Claim 164 (Proposition VI.24: Parallelograms about the diagonal are similar to the whole)**.** In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Evidence 164 (Proof of VI.24)**.** Lines drawn parallel to the sides through a point on the diagonal make the inner parallelograms equiangular with the whole (I.29) and with corresponding sides cut in the same ratio (VI.2); hence similar (VI.4).

Claim 165 (Proposition VI.25: Construct a figure similar to one and equal to another)**.** To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure.

Evidence 165 (Proof of VI.25)**.** Reduce both given figures to rectangles on a common base (I.44); the side opposite the common base measures each figure's area. Take the mean proportional (VI.13) of those two opposite sides; build on that mean a figure similar to the first via VI.18. By VI.20 the constructed figure has the required area.

Claim 166 (Proposition VI.26: Inner parallelogram about the diagonal must share a corner)**.** If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole.

Evidence 166 (Proof of VI.26)**.** Reductio: if the inner parallelogram is not about the diagonal, then VI.24 forces a similar parallelogram on the actual diagonal that differs from the given one; matching corresponding sides via the similarity contradicts the assumed common-angle configuration.

Claim 167 (Proposition VI.27: Maximum-area parallelogram with deficiency)**.** Of all parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described upon the half of the straight line, the greatest is that which is applied to the half and is similar to the deficient figure.

Evidence 167 (Proof of VI.27)**.** Comparison of areas via VI.20 and VI.24: the application to the half exhausts the bound, while any other application produces a smaller parallelogram by a square-on-the-deviation.

Claim 168 (Proposition VI.28: Apply a parallelogram with deficiency)**.** To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one; thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.

Evidence 168 (Proof of VI.28)**.** Construct the application by VI.25 to match the given figure, then verify the bound via VI.27. The construction effectively solves a quadratic equation in geometric form.

Claim 169 (Proposition VI.29: Apply a parallelogram with excess)**.** To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one.

Evidence 169 (Proof of VI.29)**.** Dual to VI.28; solves the geometric quadratic with the opposite sign.

Claim 170 (Proposition VI.30: Golden section by application of areas)**.** To cut a given finite straight line in extreme and mean ratio.

Evidence 170 (Proof of VI.30)**.** Apply to the given line a parallelogram equal to the square on it, exceeding by a square (VI.29). The exceeding square's side x satisfies $x(a+x) = a^2$, equivalently $a : x = x : (a-x)$ in the language of II.11 – the golden section.

Claim 171 (Proposition VI.31: Generalised Pythagoras)**.** In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Evidence 171 (Proof of VI.31)**.** By VI.8 the altitude from the right angle cuts the hypotenuse into segments such that each leg is the mean proportional between the hypotenuse and the adjacent segment. Applying VI.20 (areas of similar figures are in the duplicate ratio of corresponding sides) gives each leg-figure equal to its adjacent piece of the hypotenuse-figure. Summing the two pieces by V.24 yields the hypotenuse-figure.

Claim 172 (Proposition VI.32: Similarly placed triangles share a vertex angle)**.** If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line.

Evidence 172 (Proof of VI.32)**.** Equal angles and proportional sides (VI.6) force the triangles to share the same straight-line continuation at the meeting vertex (I.14).

Claim 173 (Proposition VI.33: Inscribed angles are as their arcs)**.** In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

Evidence 173 (Proof of VI.33)**.** Use III.27 (equal arcs subtend equal central angles in equal circles) to set up an equimultiples test: for any positive integers m, n, m copies of one angle correspond to m copies of its arc, and the order of $m\alpha$ versus $n\beta$ matches the order of $m \cdot \mathrm{arc}(\alpha)$ versus $n \cdot \mathrm{arc}(\beta)$. This is exactly Definition V.5 for $\alpha : \beta = \mathrm{arc}(\alpha) : \mathrm{arc}(\beta)$. The inscribed-angle case follows from III.20 (inscribed angle is half the central).

7 Book VII — Number Theory Foundations

Claim 174 (Proposition VII.1: Coprime detection by anthyphairesis)**.** Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.

Evidence 174 (Proof of VII.1)**.** Suppose for contradiction some number $d > 1$ measures both inputs. At each subtraction step the divisor and remainder differ from the previous pair by a common multiple of d; so d persists through the algorithm and ultimately measures the unit, which is impossible.

Claim 175 (Proposition VII.2: Greatest common measure)**.** Given two numbers not prime to one another, to find their greatest common measure.

Evidence 175 (Proof of VII.2)**.** Run the anthyphairesis (Euclidean algorithm). Since the numbers are not prime to one another, the procedure terminates at a non-unit remainder d; that d measures both inputs, and any other common measure also divides d by the same persistence argument as VII.1.

Claim 176 (Proposition VII.3: Greatest common measure of three numbers)**.** Given three numbers not prime to one another, to find their greatest common measure.

Evidence 176 (Proof of VII.3)**.** Find the gcd of two of them by VII.2; then find the gcd of that result with the third. The final number measures all three and is greatest by the same persistence argument.

Claim 177 (Proposition VII.4: Any number is a part or parts of a greater)**.** Any number is either a part or parts of any number, the less of the greater.

Evidence 177 (Proof of VII.4)**.** Either the less divides the greater (a part, Definition VII.3) or it does not (parts, Definition VII.4). The Euclidean algorithm terminates, so the case analysis is exhaustive.

Claim 178 (Proposition VII.5: Equal parts of equals are equal parts of the sum)**.** If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the other.

Evidence 178 (Proof of VII.5)**.** $a = b/n$ and $c = d/n$ give $a + c = (b + d)/n$ by gathering n copies of $a + c$ together.

Claim 179 (Proposition VII.6: Same parts of parts)**.** If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the other.

Evidence 179 (Proof of VII.6)**.** Same argument as VII.5 with p/q fractions; each fractional piece sums separately by VII.5 and then by VII.5 again on the multiples.

Claim 180 (Proposition VII.7: Difference of equal parts)**.** If a number be the same part of a number that a subtracted number is of a subtracted number, the remainder will also be the same part of the remainder that the whole is of the whole.

Evidence 180 (Proof of VII.7)**.** If $a = b/n$ and $a' = b'/n$ then $a - a' = (b - b')/n$ by Common Notion 3 applied to each of the n pieces.

Claim 181 (Proposition VII.8: Difference of equal parts (generalised))**.** If a number be the same parts of a number that a subtracted number is of a subtracted number, the remainder will also be the same parts of the remainder that the whole is of the whole.

Evidence 181 (Proof of VII.8)**.** Same as VII.7 but for p/q fractions; combine VII.7 with VII.6.

Claim 182 (Proposition VII.9: Symmetric form of parts)**.** If a number be a part of a number, and another be the same part of another, alternately, whatever part or parts the first is of the third, the same part or the same parts will the second also be of the fourth.

Evidence 182 (Proof of VII.9)**.** The "part of a part" relation is symmetric across the alternation by construction; apply VII.5 / VII.6 on a per-piece basis to confirm.

Claim 183 (Proposition VII.10: Alternation of parts)**.** If a number be parts of a number, and another be the same parts of another, alternately, whatever parts or part the first is of the third, the same parts or the same part will the second also be of the fourth.

Evidence 183 (Proof of VII.10)**.** Same scheme as VII.9 with p/q fractions.

Claim 184 (Proposition VII.11: Subtraction in a proportion)**.** If, as whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole is to whole.

Evidence 184 (Proof of VII.11)**.** Discrete analogue of V.19: subtract equimultiples on antecedents and consequents and the proportion is preserved.

Claim 185 (Proposition VII.12: Sum of antecedents to sum of consequents)**.** If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Evidence 185 (Proof of VII.12)**.** Discrete analogue of V.12: each pair contributes the same multiple or part, so the sum does too.

Claim 186 (Proposition VII.13: Alternation)**.** If four numbers be proportional, they will also be proportional alternately.

Evidence 186 (Proof of VII.13)**.** Discrete analogue of V.16, deduced from VII.9 / VII.10 (the parts versions of alternation).

Claim 187 (Proposition VII.14: Ex aequali for numbers)**.** If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio ex aequali.

Evidence 187 (Proof of VII.14)**.** Discrete analogue of V.22: chain alternation (VII.13) through the intermediate ratios.

Claim 188 (Proposition VII.15: Multiplying preserves ratio with unit)**.** If a unit measure any number, and another number measure any other number the same number of times, then, alternately, the unit will measure the third number the same number of times as the second measures the fourth.

Evidence 188 (Proof of VII.15)**.** Discrete analogue of V.7 / V.16 applied to the unit; the unit relations transfer through the equimultiples.

Claim 189 (Proposition VII.16: Commutativity of multiplication)**.** If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

Evidence 189 (Proof of VII.16)**.** $a \cdot b$ counts the units in a b-many sum of a's; $b \cdot a$ counts the units in an a-many sum of b's. Both equal ab by VII.5 applied to the unit.

Claim 190 (Proposition VII.17: Distributivity over equal multiples)**.** If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied.

Evidence 190 (Proof of VII.17)**.** $c \cdot a : c \cdot b = a : b$ by gathering the c copies of each side; the equimultiples test of definition V.5 (in its discrete specialisation, VII.20) reduces to the test on (a, b).

Claim 191 (Proposition VII.18: Multiplication on the consequent)**.** If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers.

Evidence 191 (Proof of VII.18)**.** $a \cdot c : b \cdot c = a : b$ by VII.16 (swap to put c on the left) and VII.17.

Claim 192 (Proposition VII.19: Equal products iff proportional numbers)**.** If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Evidence 192 (Proof of VII.19)**.** Discrete analogue of VI.16: cross-multiplication preserves and detects proportionality. Use VII.17 in one direction and the converse argument (proportion from equal products via VII.14) in the other.

Claim 193 (Proposition VII.20: Least numbers in a ratio measure the others)**.** The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

Evidence 193 (Proof of VII.20)**.** Given $a : b = p : q$ with p, q least in their ratio, by VII.2 $\gcd(a, b) \cdot p = a$ and $\gcd(a, b) \cdot q = b$, so p, q each measure a, b the same number of times $\gcd(a, b)$.

Claim 194 (Proposition VII.21: Numbers prime to one another are least in their ratio)**.** Numbers prime to one another are the least of those which have the same ratio with them.

Evidence 194 (Proof of VII.21)**.** If two coprime numbers a, b had smaller numbers c, d with $a : b = c : d$, then by VII.20 c, d would measure a, b a common number of times; the common measure would contradict coprimality.

Claim 195 (Proposition VII.22: Least in a ratio are coprime)**.** The least numbers of those which have the same ratio with them are prime to one another.

Evidence 195 (Proof of VII.22)**.** Converse of VII.21: any common measure of the least pair would generate a strictly smaller pair in the same ratio, contradicting minimality.

Claim 196 (Proposition VII.23: Coprime to a product)**.** If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.

Evidence 196 (Proof of VII.23)**.** If a, b are coprime and $d \mid a$, any common divisor of d and b would divide both a and b, contradicting coprimality.

Claim 197 (Proposition VII.24: Coprime preserved under multiplication)**.** If two numbers be prime to any number, their product also will be prime to the same.

Evidence 197 (Proof of VII.24)**.** If $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, then any prime dividing ab and c must divide a or b (using the descent from VII.23) and would contradict one of the hypotheses.

Claim 198 (Proposition VII.25: Coprime preserved under squaring)**.** If two numbers be prime to one another, the product of one of them into itself will be prime to the remaining one.

Evidence 198 (Proof of VII.25)**.** Specialise VII.24 with $b = a$.

Claim 199 (Proposition VII.26: Product of coprime pairs is coprime)**.** If two numbers be prime to two numbers, both to each, their products also will be prime to one another.

Evidence 199 (Proof of VII.26)**.** $\gcd(ab, cd) = 1$ when $\gcd(a, c) = \gcd(a, d) = \gcd(b, c) = \gcd(b, d) = 1$: apply VII.24 twice.

Claim 200 (Proposition VII.27: Coprime preserved under arbitrary powers)**.** If two numbers be prime to one another, and each by multiplying itself make a certain number, the products will be prime to one another; and if the original numbers by multiplying the products make certain numbers, these will be prime to one another.

Evidence 200 (Proof of VII.27)**.** Iterate VII.25: $\gcd(a, b) = 1$ implies $\gcd(a^n, b^m) = 1$ for all n, m.

Claim 201 (Proposition VII.28: Sum coprime iff each summand coprime to a third)**.** If two numbers be prime to one another, the sum will also be prime to each of them; and if the sum of two numbers be prime to either of them, the original numbers will also be prime to one another.

Evidence 201 (Proof of VII.28)**.** Any common divisor of $a + b$ and a would also divide b (by Common Notion 3), contradicting $\gcd(a, b) = 1$. Converse runs the same way.

Claim 202 (Proposition VII.29: Prime is coprime to anything it does not measure)**.** Any prime number is prime to any number which it does not measure.

Evidence 202 (Proof of VII.29)**.** The only divisors of a prime are 1 and itself; if the prime does not divide the other number, the only common divisor is 1.

Claim 203 (Proposition VII.30: Euclid's lemma)**.** If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

Evidence 203 (Proof of VII.30)**.** If a prime $p \mid ab$ but $p \nmid a$, then by VII.29 p is coprime to a; by VII.24 (taking b as the multiplicand) p must divide b.

Claim 204 (Proposition VII.31: Every number has a prime divisor)**.** Any composite number is measured by some prime number.

Evidence 204 (Proof of VII.31)**.** Descend through divisors: a composite has a proper divisor, that divisor is either prime or composite; if composite, repeat. The descent must terminate (numbers are bounded below by 2), so a prime divisor exists.

Claim 205 (Proposition VII.32: Every number is prime or has a prime divisor)**.** Any number either is prime or is measured by some prime number.

Evidence 205 (Proof of VII.32)**.** Case split: if the number is prime, done; otherwise apply VII.31.

Claim 206 (Proposition VII.33: Reduction to least numbers in same ratio)**.** Given as many numbers as we please, to find the least of those which have the same ratio with them.

Evidence 206 (Proof of VII.33)**.** Divide each given number by their common gcd (VII.3); the quotients are pairwise coprime (VII.22) and least in the original ratio.

Claim 207 (Proposition VII.34: Least common multiple of two numbers)**.** Given two numbers, to find the least number which they measure.

Evidence 207 (Proof of VII.34)**.** $\text{lcm}(a, b) = ab/\gcd(a, b)$: multiply a by b then divide by the gcd (VII.2).

Claim 208 (Proposition VII.35: Common multiple is a multiple of the lcm)**.** If two numbers measure any number, the least number measured by them will also measure the same.

Evidence 208 (Proof of VII.35)**.** Let $L = \text{lcm}(a, b)$ and suppose $a, b \mid n$. Divide n by L with remainder; the remainder r is measured by both a and b (Common Notion 3 on the equimultiples) and is less than L. By minimality of L, $r = 0$.

Claim 209 (Proposition VII.36: Least common multiple of three numbers)**.** Given three numbers, to find the least number which they measure.

Evidence 209 (Proof of VII.36)**.** Compute $\text{lcm}(a, b)$ by VII.34, then lcm of that with c. By VII.35 the result is the least common multiple of all three.

Claim 210 (Proposition VII.37: Number measures iff it is a quotient)**.** If a number be measured by any number, the number which is measured will have a part called by the same name as the measuring number.

Evidence 210 (Proof of VII.37)**.** $a \mid b$ means $b = ka$ for some k; then $b/k = a$, i.e. the k-th part of b is a.

Claim 211 (Proposition VII.38: Quotient existence)**.** If a number have any part whatever, it will be measured by a number called by the same name as the part.

Evidence 211 (Proof of VII.38)**.** Converse of VII.37: if $b/k = a$ then $a \mid b$ with quotient k.

Claim 212 (Proposition VII.39: Least number with prescribed parts)**.** To find the number which is the least that will have given parts.

Evidence 212 (Proof of VII.39)**.** Take the lcm (VII.34, VII.36) of the prescribed denominators; that is the smallest number containing all the prescribed parts.

8 Book VIII — Continued Proportions

Claim 213 (Proposition VIII.1: Least continued proportion has coprime extremes)**.** If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Evidence 213 (Proof of VIII.1)**.** Suppose a smaller set $b_1, \ldots, b_n$ in the same ratio existed. By ex aequali (VII.14) the ratio of extremes $b_1 : b_n$ equals $a_1 : a_n$, so by VII.21 (least in a ratio are coprime) the original a_1, a_n would not be coprime — contradiction.

Claim 214 (Proposition VIII.2: Construct a continued proportion with a given common ratio)**.** To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.

Evidence 214 (Proof of VIII.2)**.** For $a : b$ with $\gcd(a, b) = 1$, the sequence $a^{n-1}, a^{n-2}b, \ldots, b^{n-1}$ is in continued proportion with ratio $a : b$; by VII.27 the extremes are coprime, so by VIII.1 the sequence is least in its ratio.

Claim 215 (Proposition VIII.3: Least continued proportion has coprime extremes (converse setup))**.** If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

Evidence 215 (Proof of VIII.3). Suppose the extremes shared a common divisor $d > 1$. By VII.20 each term would be divisible by some power of d, producing a smaller sequence in the same ratio — contradicting minimality.

Claim 216 (Proposition VIII.4: Common ratio across multiple chains). Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Evidence 216 (Proof of VIII.4). Reduce each ratio to lowest terms by VII.33. Compound them by multiplying numerators and denominators across; the resulting sequence is in continued proportion with the prescribed ratios.

Claim 217 (Proposition VIII.5: Plane numbers have a compound ratio of their sides). Plane numbers have to one another the ratio compounded of the ratios of their sides.

Evidence 217 (Proof of VIII.5). For plane numbers ab and cd: $ab : cd = (a : c) \cdot (b : d)$ in the language of compound ratios. Verified by direct computation using VII.17 / VII.18.

Claim 218 (Proposition VIII.6: First measures last iff first measures second). If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other.

Evidence 218 (Proof of VIII.6). Contrapositive of VIII.7: divisibility propagates through the sequence, so failure at the first step prevents any later divisibility relation.

Claim 219 (Proposition VIII.7: First measures last implies first measures second). If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also.

Evidence 219 (Proof of VIII.7). If $a_1 \mid a_n$, reduce $a_1, \ldots, a_n$ to lowest terms (VIII.3); since the lowest extremes are coprime but a_1 divides a_n, the ratio $a_1 : a_n$ must be 1:1 in lowest terms, forcing $a_1 \mid a_2$ via VII.20.

Claim 220 (Proposition VIII.8: Intermediate numbers in a proportion). If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.

Evidence 220 (Proof of VIII.8). The number of geometric means between two numbers depends only on their ratio; scaling by a common factor changes the magnitudes but not the ratio, so the same number of means fall between the scaled pair.

Claim 221 (Proposition VIII.9: Coprimality between unit and a sequence). If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between each of them and a unit.

Evidence 221 (Proof of VIII.9). Coprime extremes correspond to a least continued proportion (VIII.1); the unit extends the proportion at both ends, and VIII.2 gives the matching extension on each side.

Claim 222 (Proposition VIII.10: Counting means between unit and number). If numbers fall between each of two numbers and a unit in continued proportion, however many numbers fall between each of them and a unit in continued proportion, so many also will fall between them in continued proportion.

Evidence 222 (Proof of VIII.10)**.** Concatenate two unit-anchored continued proportions; VII.14 (ex aequali) confirms the joined sequence remains in continued proportion.

Claim 223 (Proposition VIII.11: Squares have one mean proportional)**.** Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

Evidence 223 (Proof of VIII.11)**.** For squares a^2, b^2: the mean proportional is ab (since $a^2 : ab = ab : b^2 = a : b$), and $a^2 : b^2$ is the duplicate of $a : b$.

Claim 224 (Proposition VIII.12: Cubes have two mean proportionals)**.** Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

Evidence 224 (Proof of VIII.12)**.** For cubes a^3, b^3: the two means are a^2b and ab^2, and $a^3 : b^3$ is the triplicate of $a : b$.

Claim 225 (Proposition VIII.13: Powers of a continued proportion are in continued proportion)**.** If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional; and if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Evidence 225 (Proof of VIII.13)**.** Squares (and cubes) of terms in continued proportion are themselves in continued proportion, by VII.27 and VIII.2.

Claim 226 (Proposition VIII.14: Square measures square iff side measures side)**.** If a square measure a square, the side will also measure the side; and if the side measure the side, the square will also measure the square.

Evidence 226 (Proof of VIII.14)**.** $a^2 \mid b^2 \iff a \mid b$, by Euclid's lemma (VII.30) applied prime-by-prime.

Claim 227 (Proposition VIII.15: Cube measures cube iff side measures side)**.** If a cube number measure a cube number, the side will also measure the side; and if the side measure the side, the cube will also measure the cube.

Evidence 227 (Proof of VIII.15)**.** Same prime-by-prime argument as VIII.14 for cubes.

Claim 228 (Proposition VIII.16: Squares non-measuring)**.** If a square measure not a square, neither will the side measure the side; and if the side measure not the side, neither will the square measure the square.

Evidence 228 (Proof of VIII.16)**.** Contrapositive of VIII.14.

Claim 229 (Proposition VIII.17: Cubes non-measuring)**.** If a cube number measure not a cube number, neither will the side measure the side; and if the side measure not the side, neither will the cube measure the cube.

Evidence 229 (Proof of VIII.17)**.** Contrapositive of VIII.15.

Claim 230 (Proposition VIII.18: Mean proportional between similar plane numbers)**.** Between two similar plane numbers there is one mean proportional number, and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Evidence 230 (Proof of VIII.18)**.** For similar plane numbers ab and cd with $a : b = c : d$, the mean proportional is the geometric mean of ab and cd, which by VII.19 / VIII.2 equals ad (or bc, equal by VII.19).

Claim 231 (Proposition VIII.19: Mean proportionals between similar solid numbers)**.** Between two similar solid numbers there fall two mean proportional numbers, and the solid number has to the solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

Evidence 231 (Proof of VIII.19)**.** For similar solid numbers abc and def: the two means are abf and aef (or symmetric variants); together they give a continued proportion in triplicate ratio.

Claim 232 (Proposition VIII.20: Mean proportional characterises similar plane numbers)**.** If one mean proportional number fall between two numbers, the numbers will be similar plane numbers.

Evidence 232 (Proof of VIII.20)**.** Converse of VIII.18: if $a : m = m : b$ then a and b admit factorisations as similar plane numbers via VII.19.

Claim 233 (Proposition VIII.21: Two mean proportionals characterise similar solid numbers)**.** If two mean proportional numbers fall between two numbers, the numbers are similar solid numbers.

Evidence 233 (Proof of VIII.21)**.** Converse of VIII.19.

Claim 234 (Proposition VIII.22: Squares in continued proportion)**.** If three numbers be in continued proportion, and the first be square, the third will also be square.

Evidence 234 (Proof of VIII.22)**.** If $a^2 : m = m : c$, then $m^2 = a^2 c$ so $c = (m/a)^2$, hence c is square. VII.19 ensures the division produces an integer.

Claim 235 (Proposition VIII.23: Cubes in continued proportion)**.** If four numbers be in continued proportion, and the first be cube, the fourth will also be cube.

Evidence 235 (Proof of VIII.23)**.** Same scheme as VIII.22 with two means; the fourth term is the cube of the ratio's denominator scaled appropriately.

Claim 236 (Proposition VIII.24: Square ratio implies square ratio)**.** If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square.

Evidence 236 (Proof of VIII.24)**.** By VIII.11 the ratio of squares has a mean proportional; transferring that mean to $a^2 : b$ forces b to be square by VIII.22.

Claim 237 (Proposition VIII.25: Cube ratio implies cube ratio)**.** If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will also be cube.

Evidence 237 (Proof of VIII.25)**.** Same argument as VIII.24 for cubes via VIII.12 and VIII.23.

Claim 238 (Proposition VIII.26: Similar plane numbers have square ratio)**.** Similar plane numbers have to one another the ratio which a square number has to a square number.

Evidence 238 (Proof of VIII.26)**.** By VIII.18 similar plane numbers admit a mean proportional, and the ratio (squares of corresponding sides) is a square-to-square ratio.

Claim 239 (Proposition VIII.27: Similar solid numbers have cube ratio)**.** Similar solid numbers have to one another the ratio which a cube number has to a cube number.

Evidence 239 (Proof of VIII.27)**.** By VIII.19 similar solid numbers admit two mean proportionals, and the ratio is in the triplicate (cube-to-cube) ratio of corresponding sides.

9 Book IX — Advanced Number Theory

Claim 240 (Proposition IX.1: Product of similar plane numbers is square)**.** If two similar plane numbers by multiplying one another make some number, the product will be square.

Evidence 240 (Proof of IX.1)**.** Similar plane numbers ab and cd with $a : b = c : d$ have product $(ab)(cd) = (ac)(bd)$, and the side $ac : bd$ is the square ratio, so the product is a square (VIII.26).

Claim 241 (Proposition IX.2: Square product implies similar plane factors)**.** If two numbers by multiplying one another make a square number, they are similar plane numbers.

Evidence 241 (Proof of IX.2)**.** Converse of IX.1. If mn is square then $m : n$ has a mean proportional in integers; by VIII.20 this means m, n are similar plane.

Claim 242 (Proposition IX.3: Cube times itself is a cube)**.** If a cube number by multiplying itself make some number, the product will be cube.

Evidence 242 (Proof of IX.3)**.** $(a^3)^2 = a^6 = (a^2)^3$. Verified via VII.16 / VII.17.

Claim 243 (Proposition IX.4: Cube times cube is cube)**.** If a cube number by multiplying a cube number make some number, the product will be cube.

Evidence 243 (Proof of IX.4)**.** $a^3 \cdot b^3 = (ab)^3$ by commutativity (VII.16).

Claim 244 (Proposition IX.5: Cube product implies cube factor)**.** If a cube number by multiplying any number make a cube number, the multiplied number will also be cube.

Evidence 244 (Proof of IX.5)**.** If $a^3 \cdot n = m^3$ then $n = m^3/a^3$; by VIII.25 the quotient of two cubes (when integer) is a cube.

Claim 245 (Proposition IX.6: Square root of a square product)**.** If a number by multiplying itself make a cube number, it itself will also be cube.

Evidence 245 (Proof of IX.6)**.** If $n^2 = a^3$ then $n^2 \cdot n = a^3 \cdot n$ is cubed; by VIII.25 n is cube.

Claim 246 (Proposition IX.7: Product of a composite with any number)**.** If a composite number by multiplying any number make some number, the product will be solid.

Evidence 246 (Proof of IX.7)**.** A composite has a factorisation ab; multiplying by c gives the triple product abc, which is solid by definition (VII.17).

Claim 247 (Proposition IX.8: Geometric progression starting from unit)**.** If as many numbers as we please beginning from a unit be in continued proportion, the third from the unit will be square, the fourth a cube, and so on.

Evidence 247 (Proof of IX.8)**.** The sequence $1, r, r^2, r^3, \ldots$ has third term a square, fourth a cube, sixth both square and cube, by VIII.11 / VIII.12.

Claim 248 (Proposition IX.9: Sixth term from unit is square and cube)**.** If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be square, all the rest will also be square; and if the number after the unit be cube, all the rest will also be cube.

Evidence 248 (Proof of IX.9)**.** In $1, r, r^2, r^3, \ldots$, if r is square then every r^k is square; if r is cube then every r^k is cube — by VIII.22 / VIII.23 propagated through the sequence.

Claim 249 (Proposition IX.10: Non-square first term keeps the sequence non-square)**.** If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be not square, neither will any other be square except the third from the unit and all those which leave out one.

Evidence 249 (Proof of IX.10)**.** Squares in $1, r, r^2, \ldots$ occur exactly at even powers; if r is not square, only r^{2k} terms are square.

Claim 250 (Proposition IX.11: Term divides later term iff exponents differ)**.** If as many numbers as we please beginning from a unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers.

Evidence 250 (Proof of IX.11)**.** $r^a \mid r^b$ iff $a \le b$, and the quotient is r^{b-a} — which is itself a term of the sequence.

Claim 251 (Proposition IX.12: Prime divisor of last term divides second)**.** If as many numbers as we please beginning from a unit be in continued proportion, by whatever prime numbers the last is measured, the second from the unit will also be measured by the same.

Evidence 251 (Proof of IX.12)**.** A prime dividing r^n must divide r (Euclid's lemma, VII.30), which is the second term after the unit.

Claim 252 (Proposition IX.13: Geometric progression from prime unit)**.** If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be prime, the greatest will not be measured by any except those which have a place among the proportional numbers.

Evidence 252 (Proof of IX.13)**.** If r is prime, the divisors of r^n are exactly $1, r, r^2, \ldots, r^n$ (Euclid's lemma). These are exactly the prefix of the sequence.

Claim 253 (Proposition IX.14: Unique prime factorisation precursor)**.** If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

Evidence 253 (Proof of IX.14)**.** A least common multiple of primes equals their product; any prime dividing the product divides one of the factors (VII.30), hence is one of the originals. This is the kernel of unique factorisation.

Claim 254 (Proposition IX.15: Three numbers in continued proportion, coprime extremes)**.** If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

Evidence 254 (Proof of IX.15)**.** For a, b, c with $a : b = b : c$ in lowest terms: by VIII.3 the extremes are coprime; combining with VII.28 the sums $a + b$, $b + c$, $a + c$ are each coprime to the third term.

Claim 255 (Proposition IX.16: Coprime numbers and proportion)**.** If two numbers be prime to one another, the second will not be to any other number as the first is to the second.

Evidence 255 (Proof of IX.16)**.** If $a : b = b : c$ with $\gcd(a, b) = 1$, then $b^2 = ac$, forcing $a \mid b^2$ which (Euclid's lemma) forces $a \mid b$ – contradicting coprimality unless $a = 1$.

Claim 256 (Proposition IX.17: Coprime sequence and extension)**.** If as many numbers as we please be in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first is to the second.

Evidence 256 (Proof of IX.17)**.** Generalisation of IX.16: a coprime-extremes proportion cannot be extended further while remaining a proportion of integers in the same ratio.

Claim 257 (Proposition IX.18: Existence of a third proportional)**.** Given two numbers, to investigate whether it is possible to find a third proportional to them.

Evidence 257 (Proof of IX.18)**.** A third proportional to a, b exists iff b^2 is divisible by a (i.e. b^2/a is an integer); apply VII.19.

Claim 258 (Proposition IX.19: Existence of a fourth proportional)**.** Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Evidence 258 (Proof of IX.19)**.** A fourth proportional to a, b, c exists iff bc/a is an integer; otherwise no integer extends the proportion.

Claim 259 (Proposition IX.20: There are infinitely many primes)**.** Prime numbers are more than any assigned multitude of prime numbers.

Evidence 259 (Proof of IX.20)**.** Given primes $p_1, \ldots, p_n$, form $N = p_1 p_2 \cdots p_n + 1$. By VII.31 N has a prime divisor q. If q were one of the p_i, then q would divide $N - p_1 \cdots p_n = 1$ (Common Notion 3), which is impossible. Hence q is a new prime not in the original list; the list of primes admits no upper bound.

Claim 260 (Proposition IX.21: Sum of evens is even)**.** If as many even numbers as we please be added together, the whole is even.

Evidence 260 (Proof of IX.21)**.** Each even number is divisible by 2; the sum is therefore divisible by 2 (Common Notion 2).

Claim 261 (Proposition IX.22: Sum of even-many odd is even)**.** If as many odd numbers as we please be added together, and their multitude be even, the whole will be even.

Evidence 261 (Proof of IX.22)**.** Pair the odd summands: each pair has even sum (since odd + odd = even, by the definitions); the total of even-many pairs is even by IX.21.

Claim 262 (Proposition IX.23: Sum of odd-many odd is odd)**.** If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd.

Evidence 262 (Proof of IX.23)**.** Group all but one of the summands by pairs (sum of those is even, IX.22); add the remaining odd number; the result is even + odd = odd by Common Notion 2 and Definition VII.7.

Claim 263 (Proposition IX.24: Difference of two evens is even)**.** If from an even number an even number be subtracted, the remainder will be even.

Evidence 263 (Proof of IX.24)**.** Two multiples of 2 differ by a multiple of 2 (Common Notion 3).

Claim 264 (Proposition IX.25: Even minus odd is odd)**.** If from an even number an odd number be subtracted, the remainder will be odd.

Evidence 264 (Proof of IX.25)**.** $2k - (2m + 1) = 2(k - m) - 1$, which is odd by Definition VII.7.

Claim 265 (Proposition IX.26: Odd minus odd is even)**.** If from an odd number an odd number be subtracted, the remainder will be even.

Evidence 265 (Proof of IX.26)**.** $(2k + 1) - (2m + 1) = 2(k - m)$, even.

Claim 266 (Proposition IX.27: Odd minus even is odd)**.** If from an odd number an even number be subtracted, the remainder will be odd.

Evidence 266 (Proof of IX.27)**.** $(2k + 1) - 2m = 2(k - m) + 1$, odd.

Claim 267 (Proposition IX.28: Odd times even is even)**.** If an odd number by multiplying an even number make some number, the product will be even.

Evidence 267 (Proof of IX.28)**.** $(2k+1) \cdot 2m = 2m(2k+1)$, a multiple of 2.

Claim 268 (Proposition IX.29: Odd times odd is odd)**.** If an odd number by multiplying an odd number make some number, the product will be odd.

Evidence 268 (Proof of IX.29)**.** $(2k+1)(2m+1) = 4km + 2k + 2m + 1$, which is odd (one more than even).

Claim 269 (Proposition IX.30: Odd dividing an even number)**.** If an odd number measure an even number, it will also measure the half of it.

Evidence 269 (Proof of IX.30)**.** If a (odd) divides $2b$, then since $\gcd(a, 2) = 1$, $a \mid b$ by VII.24.

Claim 270 (Proposition IX.31: Odd coprime to its double)**.** If an odd number be prime to any number, it will also be prime to the double of it.

Evidence 270 (Proof of IX.31)**.** $\gcd(a, n) = 1$ for a odd implies $\gcd(a, 2n) = 1$ (since $\gcd(a, 2) = 1$); apply VII.24.

Claim 271 (Proposition IX.32: Powers of 2 are even-times even)**.** Each of the numbers which are continually doubled beginning from a duad is even-times even only.

Evidence 271 (Proof of IX.32)**.** 2^n has no odd divisors greater than 1 (Euclid's lemma); hence its only factorisation is $2 \times 2^{n-1}$, even-times-even.

Claim 272 (Proposition IX.33: Number whose half is odd)**.** If a number have its half odd, it is even-times odd only.

Evidence 272 (Proof of IX.33)**.** $n = 2k$ with k odd is even-times-odd; it cannot be even-times-even because then its half would be even.

Claim 273 (Proposition IX.34: Neither power of 2 nor twice-odd: both kinds)**.** If an even number be neither one of those which are doubled from a duad, nor have its half odd, it is both even-times even and even-times odd.

Evidence 273 (Proof of IX.34)**.** Such a number has the form $2^k m$ with $k \geq 2$ and m odd, $m > 1$: it is even-times even $(2 \cdot 2^{k-1}m)$ and even-times odd $(2^k \cdot m)$.

Claim 274 (Proposition IX.35: Geometric series formula)**.** If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

Evidence 274 (Proof of IX.35)**.** For $a, ar, ar^2, \ldots, ar^n$: the excess of the last over the first is $a(r^n - 1)$, and the sum of the rest is $a(1 + r + r^2 + \cdots + r^{n-1}) = a(r^n - 1)/(r - 1)$; the ratio of excess to sum equals $(r - 1) = (ar - a)/a$, the excess of the second over the first divided by the first.

Claim 275 (Proposition IX.36: Even perfect number theorem)**.** If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

Evidence 275 (Proof of IX.36)**.** Let $s = 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$ be prime (a Mersenne prime); set $N = s \cdot 2^{n-1}$. The proper divisors of N are $1, 2, 4, \ldots, 2^{n-1}, s, 2s, \ldots, 2^{n-2}s$, whose sum is $(2^n - 1) + s(2^{n-1} - 1) = s + s(2^{n-1} - 1) = s \cdot 2^{n-1} = N$. So N equals the sum of its proper divisors and is perfect.

10 Book X — Incommensurable Magnitudes

Claim 276 (Proposition X.1: Method of exhaustion lemma)**.** Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Evidence 276 (Proof of X.1)**.** By the Archimedean property (Definition V.4), some multiple of the smaller magnitude exceeds the larger. Iterated halving (or more) brings the remainder below the smaller in a finite number of steps.

Claim 277 (Proposition X.2: Anthyphairesis detects incommensurability)**.** If, when the lesser of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

Evidence 277 (Proof of X.2)**.** A common measure would persist through anthyphairesis (Common Notion 3); non-termination of the algorithm thus implies no common measure exists.

Claim 278 (Proposition X.3: Greatest common measure of commensurables)**.** Given two commensurable magnitudes, to find their greatest common measure.

Evidence 278 (Proof of X.3)**.** Apply anthyphairesis (the Euclidean algorithm on magnitudes); by X.2 the algorithm terminates exactly when a common measure exists.

Claim 279 (Proposition X.4: GCM of three commensurables)**.** Given three commensurable magnitudes, to find their greatest common measure.

Evidence 279 (Proof of X.4)**.** Apply X.3 to the first two; then to the result with the third.

Claim 280 (Proposition X.5: Commensurables have a number ratio)**.** Commensurable magnitudes have to one another the ratio which a number has to a number.

Evidence 280 (Proof of X.5)**.** If d is a common measure of a, b, then $a = md$, $b = nd$, so $a : b = m : n$ (a ratio of integers).

Claim 281 (Proposition X.6: Number ratio implies commensurable)**.** If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

Evidence 281 (Proof of X.6)**.** Converse of X.5: if $a : b = m : n$, dividing a by m produces a common measure of a and b.

Claim 282 (Proposition X.7: Incommensurables have no number ratio)**.** Incommensurable magnitudes have not to one another the ratio which a number has to a number.

Evidence 282 (Proof of X.7)**.** Contrapositive of X.6.

Claim 283 (Proposition X.8: No number ratio implies incommensurable)**.** If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable.

Evidence 283 (Proof of X.8)**.** Contrapositive of X.5.

Claim 284 (Proposition X.9: Commensurability of squares from commensurability of sides)**.** The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length.

Evidence 284 (Proof of X.9)**.** If $a : b = m : n$ (integers) then $a^2 : b^2 = m^2 : n^2$; converse holds by VIII.14 applied to integers and VI.22 applied to magnitudes.

Claim 285 (Proposition X.10: Construct incommensurables)**.** To find two straight lines incommensurable, the one in length only, the other in square also, with an assigned straight line.

Evidence 285 (Proof of X.10)**.** Take a non-square integer ratio (e.g. $2 : 1$) and use it via X.9 to construct an incommensurable-in-length pair; build the incommensurable-in-square one similarly with a ratio that is neither square nor cube.

Claim 286 (Proposition X.11: Commensurability is transitive)**.** If four magnitudes be proportional, and the first be commensurable with the second, the third also will be commensurable with the fourth; and if the first be incommensurable with the second, the third also will be incommensurable with the fourth.

Evidence 286 (Proof of X.11)**.** Commensurability $\iff$ rational ratio (X.5 / X.6); rational ratios are preserved under equality of ratios.

Claim 287 (Proposition X.12: Commensurability is transitive (three magnitudes))**.** Magnitudes commensurable with the same magnitude are commensurable with one another.

Evidence 287 (Proof of X.12)**.** If $a \sim c$ and $b \sim c$ (commensurable), then $a \sim b$ by composing ratios (X.11).

Claim 288 (Proposition X.13: Incommensurable preserved through transitivity)**.** If two magnitudes be commensurable, and one of them be incommensurable with any magnitude, the remaining one will also be incommensurable with the same.

Evidence 288 (Proof of X.13)**.** Contrapositive of X.12.

Claim 289 (Proposition X.14: Squares preserve commensurability of sides)**.** If four straight lines be proportional, and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third.

Evidence 289 (Proof of X.14)**.** Proportionality lifts to the squares; the deviation magnitude inherits the commensurability relation.

Claim 290 (Proposition X.15: Sum of commensurables is commensurable)**.** If two commensurable magnitudes be added together, the whole will also be commensurable with each of them; and if the whole be commensurable with one of them, the original magnitudes will also be commensurable.

Evidence 290 (Proof of X.15)**.** a, b have a common measure d, so $a + b = (m + n)d$ shares d. Converse: if $a + b$ and a share a measure, then so does b by Common Notion 3.

Claim 291 (Proposition X.16: Sum with incommensurable)**.** If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

Evidence 291 (Proof of X.16)**.** Contrapositive of X.15.

Claim 292 (Proposition X.17: Application of areas with commensurable difference)**.** If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are commensurable in length, then the square on the greater will be greater than the square on the less by the square on a straight line commensurable in length with the greater.

Evidence 292 (Proof of X.17)**.** By VI.28 / VI.29 the application of areas with deficient/excess square corresponds to solving a quadratic; commensurability of the parts forces commensurability of the discriminant.

Claim 293 (Proposition X.18: Application of areas with incommensurable difference)**.** If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts incommensurable in length, then the square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater.

Evidence 293 (Proof of X.18)**.** Same construction as X.17 with the opposite hypothesis; incommensurability of the parts forces incommensurability of the discriminant.

Claim 294 (Proposition X.19: Rectangle on rationals is rational)**.** The rectangle contained by rational straight lines commensurable in length is rational.

Evidence 294 (Proof of X.19)**.** Rational sides commensurable in length have integer ratios; product of two such sides is in rational ratio to the assigned-square area.

Claim 295 (Proposition X.20: Rational area divided by rational side is rational)**.** If a rational area be applied to a rational straight line, it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.

Evidence 295 (Proof of X.20)**.** $A = \ell \cdot b$ with A and ℓ rational forces b rational by X.19 / X.9.

Claim 296 (Proposition X.21: Medial rectangle)**.** The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial.

Evidence 296 (Proof of X.21)**.** Commensurable-in-square-only means the square on each is rational but the lengths are not in integer ratio. The rectangle is then in a non-rational ratio to a rational area; its square root is the medial straight line (Definition XIII.3).

Claim 297 (Proposition X.22: Square on a medial)**.** The square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied.

Evidence 297 (Proof of X.22)**.** A medial squared is rational; applying it to a rational base, the breadth is rational; the incommensurability follows from the fact that the medial is incommensurable in length with the rational.

Claim 298 (Proposition X.23: Magnitudes commensurable with medial are medial)**.** A straight line commensurable with a medial straight line is medial.

Evidence 298 (Proof of X.23)**.** Commensurability preserves the medial property: scaling a medial by a rational ratio leaves it medial.

Claim 299 (Proposition X.24: Rectangle of commensurable medials)**.** The rectangle contained by medial straight lines commensurable in length is medial.

Evidence 299 (Proof of X.24)**.** Product of two medials in rational length-ratio is again the geometric mean of two rationals (Definition XIII.3).

Claim 300 (Proposition X.25: Rectangle of medials commensurable in square only)**.** The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

Evidence 300 (Proof of X.25)**.** Two cases depending on whether the rectangle has a rational square-root. Both cases are realised by explicit constructions.

Claim 301 (Proposition X.26: Medial difference is not rational)**.** A medial area does not exceed a medial area by a rational area.

Evidence 301 (Proof of X.26)**.** A difference $M_1 - M_2 = R$ with R rational and M_1, M_2 medial would force M_1, M_2 to be in a rational ratio, contradicting their being medial in distinct families.

Claim 302 (Proposition X.27: Two medial lines commensurable in square)**.** To find medial straight lines commensurable in square only which contain a rational rectangle.

Evidence 302 (Proof of X.27)**.** Construct two such medials from a fixed rational by extracting two square-roots of ratios in lowest terms.

Claim 303 (Proposition X.28: Medials enclosing a medial rectangle)**.** To find medial straight lines commensurable in square only which contain a medial rectangle.

Evidence 303 (Proof of X.28)**.** Same construction as X.27 with an extra medial step.

Claim 304 (Proposition X.29: Two rationals commensurable in square, square-difference of commensurable kind, Lemma 1)**.** To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Evidence 304 (Proof of X.29)**.** Take a rational line a and apply X.17 with a deficient square; the construction gives the required pair.

Claim 305 (Proposition X.30: Same as X.29 with the discriminant incommensurable)**.** To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater.

Evidence 305 (Proof of X.30)**.** Apply X.18 (the incommensurable analogue of X.17).

Claim 306 (Proposition X.31: Two medials with rational discriminant relation)**.** To find two medial straight lines commensurable in square only, containing a rational rectangle, such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Evidence 306 (Proof of X.31)**.** Combine X.27 with the discriminant-control of X.29.

Claim 307 (Proposition X.32: Two medials, medial rectangle, commensurable discriminant)**.** To find two medial straight lines commensurable in square only, containing a medial rectangle, such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Evidence 307 (Proof of X.32)**.** Same as X.31 with X.28 in place of X.27.

Claim 308 (Proposition X.33: Sum of squares rational, rectangle medial, sides incommensurable in square)**.** To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial.

Evidence 308 (Proof of X.33)**.** Apply X.30: a difference-of-squares construction with an incommensurable discriminant produces such a pair.

Claim 309 (Proposition X.34: Sum medial, rectangle rational)**.** To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational.

Evidence 309 (Proof of X.34)**.** Variant of X.33 with the medial/rational roles swapped, via X.31.

Claim 310 (Proposition X.35: Sum medial, rectangle medial, sum incommensurable with rectangle)**.** To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Evidence 310 (Proof of X.35)**.** Variant of X.34 with both quantities medial; use X.32.

Claim 311 (Proposition X.36: Binomial straight line)**.** If two rational straight lines commensurable in square only be added together, the whole is irrational; and let it be called binomial.

Evidence 311 (Proof of X.36)**.** The sum $a + b$ with a, b rational and incommensurable in length has square $a^2 + b^2 + 2ab$ where $2ab$ is medial (X.21), so the square on the sum is the sum of a rational and a medial: irrational.

Claim 312 (Proposition X.37: First bimedial straight line)**.** If two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational; and let it be called first bimedial.

Evidence 312 (Proof of X.37)**.** Sum of two medials with rational rectangle; the square consists of two medials and a rational — irrational.

Claim 313 (Proposition X.38: Second bimedial straight line)**.** If two medial straight lines commensurable in square only and containing a medial rectangle be added together, the whole is irrational; and let it be called second bimedial.

Evidence 313 (Proof of X.38)**.** Same scheme as X.37 with the rectangle medial instead of rational.

Claim 314 (Proposition X.39: Major straight line)**.** If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole straight line is irrational; and let it be called major.

Evidence 314 (Proof of X.39)**.** Sum of an X.33 pair has square = rational + medial: irrational.

Claim 315 (Proposition X.40: Side of a rational plus medial area)**.** If two straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational, be added together, the whole straight line is irrational; and let it be called the side of a rational plus a medial area.

Evidence 315 (Proof of X.40)**.** Sum of an X.34 pair; same scheme as X.39.

Claim 316 (Proposition X.41: Side of the sum of two medial areas)**.** If two straight lines incommensurable in square which make the sum of the squares on them medial, and the rectangle contained by them medial and also incommensurable with the sum of the squares on them, be added together, the remaining straight line is irrational; and let it be called the side of the sum of two medial areas.

Evidence 316 (Proof of X.41)**.** Sum of an X.35 pair; same scheme as X.39.

Claim 317 (Proposition X.42: A binomial has unique decomposition)**.** A binomial straight line is divided into its terms at one point only.

Evidence 317 (Proof of X.42)**.** Suppose two decompositions $a_1 + b_1 = a_2 + b_2$ of the same binomial. Comparing rationals and medials in the squares forces $a_1 = a_2$ and $b_1 = b_2$.

Claim 318 (Proposition X.43: A first bimedial has unique decomposition)**.** A first bimedial straight line is divided at one and the same point only.

Evidence 318 (Proof of X.43)**.** Same uniqueness argument as X.42 applied to the first bimedial.

Claim 319 (Proposition X.44: A second bimedial has unique decomposition)**.** A second bimedial straight line is divided at one point only.

Evidence 319 (Proof of X.44)**.** Same uniqueness argument applied to the second bimedial.

Claim 320 (Proposition X.45: A major has unique decomposition)**.** A major straight line is divided at one and the same point only.

Evidence 320 (Proof of X.45)**.** Same uniqueness argument applied to the major.

Claim 321 (Proposition X.46: Side of rational+medial has unique decomposition)**.** The side of a rational plus a medial area is divided at one and the same point only.

Evidence 321 (Proof of X.46)**.** Same uniqueness argument.

Claim 322 (Proposition X.47: Side of two medial areas has unique decomposition)**.** The side of the sum of two medial areas is divided at one and the same point only.

Evidence 322 (Proof of X.47)**.** Same uniqueness argument.

Claim 323 (Proposition X.48: First binomial straight line)**.** To find the first binomial straight line.

Evidence 323 (Proof of X.48)**.** Construct $a + b$ with a commensurable in length with the assigned rational and the square-discriminant commensurable with the greater (X.29).

Claim 324 (Proposition X.49: Second binomial straight line)**.** To find the second binomial straight line.

Evidence 324 (Proof of X.49)**.** Construct as in X.48 but with b (rather than a) commensurable with the assigned rational.

Claim 325 (Proposition X.50: Third binomial straight line)**.** To find the third binomial straight line.

Evidence 325 (Proof of X.50)**.** Neither term commensurable with the assigned rational, but the square-discriminant commensurable with the greater (X.29 variant).

Claim 326 (Proposition X.51: Fourth binomial straight line)**.** To find the fourth binomial straight line.

Evidence 326 (Proof of X.51)**.** a commensurable with the assigned rational, square-discriminant incommensurable with a (X.30).

Claim 327 (Proposition X.52: Fifth binomial straight line)**.** To find the fifth binomial straight line.

Evidence 327 (Proof of X.52)**.** b commensurable, discriminant incommensurable with the greater.

Claim 328 (Proposition X.53: Sixth binomial straight line)**.** To find the sixth binomial straight line.

Evidence 328 (Proof of X.53)**.** Neither term commensurable, discriminant incommensurable.

Claim 329 (Proposition X.54: Rectangle on a first binomial is rational on rational)**.** If an area be contained by a rational straight line and the first binomial, the side of the area is the irrational straight line which is called binomial.

Evidence 329 (Proof of X.54)**.** $\sqrt{R \cdot \text{first binomial}}$ has the form of a binomial in the assigned rational base.

Claim 330 (Proposition X.55: Rectangle on a second binomial yields a first bimedial)**.** If an area be contained by a rational straight line and the second binomial, the side of the area is the irrational straight line which is called first bimedial.

Evidence 330 (Proof of X.55)**.** Same pattern as X.54.

Claim 331 (Proposition X.56: Rectangle on a third binomial yields a second bimedial)**.** If an area be contained by a rational straight line and the third binomial, the side of the area is the irrational straight line which is called second bimedial.

Evidence 331 (Proof of X.56)**.** Same pattern.

Claim 332 (Proposition X.57: Rectangle on a fourth binomial yields a major)**.** If an area be contained by a rational straight line and the fourth binomial, the side of the area is the irrational straight line which is called major.

Evidence 332 (Proof of X.57)**.** Same pattern.

Claim 333 (Proposition X.58: Rectangle on a fifth binomial yields a side of rational+medial)**.** If an area be contained by a rational straight line and the fifth binomial, the side of the area is the irrational straight line which is the side of a rational plus a medial area.

Evidence 333 (Proof of X.58)**.** Same pattern.

Claim 334 (Proposition X.59: Rectangle on a sixth binomial yields side of two medials)**.** If an area be contained by a rational straight line and the sixth binomial, the side of the area is the irrational straight line which is called the side of the sum of two medial areas.

Evidence 334 (Proof of X.59)**.** Same pattern.

Claim 335 (Proposition X.60: Square on a binomial yields a first binomial)**.** The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial.

Evidence 335 (Proof of X.60)**.** Inverse of X.54: squaring and dividing by the assigned rational recovers the first binomial.

Claim 336 (Proposition X.61: Square on a first bimedial yields a second binomial)**.** The square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial.

Evidence 336 (Proof of X.61)**.** Inverse of X.55.

Claim 337 (Proposition X.62: Square on a second bimedial yields a third binomial)**.** The square on the second bimedial straight line applied to a rational straight line produces as breadth the third binomial.

Evidence 337 (Proof of X.62)**.** Inverse of X.56.

Claim 338 (Proposition X.63: Square on a major yields a fourth binomial)**.** The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial.

Evidence 338 (Proof of X.63)**.** Inverse of X.57.

Claim 339 (Proposition X.64: Square on side-of-rational-plus-medial yields a fifth binomial)**.** The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial.

Evidence 339 (Proof of X.64)**.** Inverse of X.58.

Claim 340 (Proposition X.65: Square on side-of-two-medials yields a sixth binomial)**.** The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.

Evidence 340 (Proof of X.65)**.** Inverse of X.59.

Claim 341 (Proposition X.66: Commensurable with binomial is binomial)**.** A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order.

Evidence 341 (Proof of X.66)**.** Multiplication by a rational ratio preserves the binomial type and order.

Claim 342 (Proposition X.67: Commensurable with bimedial is bimedial)**.** A straight line commensurable in length with a bimedial straight line is itself bimedial and the same in order.

Evidence 342 (Proof of X.67)**.** Same scheme as X.66.

Claim 343 (Proposition X.68: Commensurable with major is major)**.** A straight line commensurable with a major straight line is itself major.

Evidence 343 (Proof of X.68)**.** Same scheme.

Claim 344 (Proposition X.69: Commensurable with side of rational+medial is the same)**.** A straight line commensurable with the side of a rational plus a medial area is itself such a side.

Evidence 344 (Proof of X.69)**.** Same scheme.

Claim 345 (Proposition X.70: Commensurable with side of two-medials is the same)**.** A straight line commensurable with the side of the sum of two medial areas is itself such a side.

Evidence 345 (Proof of X.70)**.** Same scheme.

Claim 346 (Proposition X.71: Rational + medial sum is one of the four irrationals)**.** If a rational and a medial area be added together, four irrational straight lines arise, namely either a binomial, a first bimedial, a major, or a side of a rational plus a medial area.

Evidence 346 (Proof of X.71)**.** The square of any of the four classes (X.36, X.37, X.39, X.40) is the sum of a rational and a medial; conversely, every such sum arises in exactly one of these forms.

Claim 347 (Proposition X.72: Medial + medial sum yields a bimedial or a side of two medials)**.** If two medial areas incommensurable with one another be added together, the remaining two irrational straight lines arise, namely either a second bimedial or a side of the sum of two medial areas.

Evidence 347 (Proof of X.72)**.** The square of X.38 or X.41 is a sum of two incommensurable medial areas; converse runs the same way.

Claim 348 (Proposition X.73: Apotome straight line)**.** If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only, the remainder is irrational; and let it be called apotome.

Evidence 348 (Proof of X.73)**.** $a - b$ with a, b commensurable in square only is the negation of the binomial case (X.36); the same argument shows it is irrational.

Claim 349 (Proposition X.74: First apotome of a medial)**.** If from a medial straight line there be subtracted a medial straight line commensurable with the whole in square only, and containing with the whole a rational rectangle, the remainder is irrational; and let it be called first apotome of a medial.

Evidence 349 (Proof of X.74)**.** Negation of X.37.

Claim 350 (Proposition X.75: Second apotome of a medial)**.** If from a medial straight line there be subtracted a medial straight line commensurable with the whole in square only, and containing with the whole a medial rectangle, the remainder is irrational; and let it be called second apotome of a medial.

Evidence 350 (Proof of X.75)**.** Negation of X.38.

Claim 351 (Proposition X.76: Minor straight line)**.** If from a straight line there be subtracted a straight line incommensurable in square with the whole, which with the whole makes the squares on them added together rational, but the rectangle contained by them medial, the remainder is irrational; and let it be called minor.

Evidence 351 (Proof of X.76)**.** Negation of X.39. This is the "minor" line (Definition XIII.4).

Claim 352 (Proposition X.77: Line producing with rational area a medial whole)**.** If from a straight line there be subtracted a straight line incommensurable in square with the whole which with the whole makes the sum of squares medial but twice the rectangle rational, the remainder is irrational; let it be called that which produces with a rational area a medial whole.

Evidence 352 (Proof of X.77)**.** Negation of X.40.

Claim 353 (Proposition X.78: Line producing with medial area a medial whole)**.** If from a straight line there be subtracted a straight line incommensurable in square with the whole which with the whole makes both the sum of squares and twice the rectangle medial and the two sums incommensurable with one another, the remainder is irrational; let it be called that which produces with a medial area a medial whole.

Evidence 353 (Proof of X.78)**.** Negation of X.41.

Claim 354 (Proposition X.79: Apotome has unique annex)**.** Only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only.

Evidence 354 (Proof of X.79)**.** Uniqueness analogue of X.42 for apotomes.

Claim 355 (Proposition X.80: First-apotome-of-medial uniqueness)**.** Only one medial straight line can be annexed to a first apotome of a medial which is commensurable with the whole in square only and forms with it a rational rectangle.

Evidence 355 (Proof of X.80)**.** Same uniqueness pattern.

Claim 356 (Proposition X.81: Second-apotome-of-medial uniqueness)**.** Only one medial straight line can be annexed to a second apotome of a medial which is commensurable with the whole in square only and forms with it a medial rectangle.

Evidence 356 (Proof of X.81)**.** Same uniqueness pattern.

Claim 357 (Proposition X.82: Minor uniqueness)**.** Only one straight line can be annexed to a minor.

Evidence 357 (Proof of X.82)**.** Same uniqueness pattern.

Claim 358 (Proposition X.83: Uniqueness for X.77's line)**.** Only one straight line can be annexed to the line producing with a rational area a medial whole.

Evidence 358 (Proof of X.83)**.** Same uniqueness pattern.

Claim 359 (Proposition X.84: Uniqueness for X.78's line)**.** Only one straight line can be annexed to the line producing with a medial area a medial whole.

Evidence 359 (Proof of X.84)**.** Same uniqueness pattern.

Claim 360 (Proposition X.85: First apotome)**.** To find the first apotome.

Evidence 360 (Proof of X.85)**.** Take a first binomial $a + b$ (X.48); the difference $a - b$ is the first apotome.

Claim 361 (Proposition X.86: Second apotome)**.** To find the second apotome.

Evidence 361 (Proof of X.86)**.** Use the second binomial as the model (X.49).

Claim 362 (Proposition X.87: Third apotome)**.** To find the third apotome.

Evidence 362 (Proof of X.87)**.** Use the third binomial (X.50).

Claim 363 (Proposition X.88: Fourth apotome)**.** To find the fourth apotome.

Evidence 363 (Proof of X.88)**.** Use the fourth binomial (X.51).

Claim 364 (Proposition X.89: Fifth apotome)**.** To find the fifth apotome.

Evidence 364 (Proof of X.89)**.** Use the fifth binomial (X.52).

Claim 365 (Proposition X.90: Sixth apotome)**.** To find the sixth apotome.

Evidence 365 (Proof of X.90)**.** Use the sixth binomial (X.53).

Claim 366 (Proposition X.91: Side of first-apotome area is an apotome)**.** If an area be contained by a rational straight line and a first apotome, the side of the area is an apotome.

Evidence 366 (Proof of X.91)**.** Dual of X.54 for apotomes.

Claim 367 (Proposition X.92: Side of second-apotome area is a first apotome of medial)**.** If an area be contained by a rational straight line and a second apotome, the side of the area is a first apotome of a medial.

Evidence 367 (Proof of X.92)**.** Dual of X.55.

Claim 368 (Proposition X.93: Side of third-apotome area is a second apotome of medial)**.** If an area be contained by a rational straight line and a third apotome, the side of the area is a second apotome of a medial.

Evidence 368 (Proof of X.93)**.** Dual of X.56.

Claim 369 (Proposition X.94: Side of fourth-apotome area is a minor)**.** If an area be contained by a rational straight line and a fourth apotome, the side of the area is a minor.

Evidence 369 (Proof of X.94)**.** Dual of X.57.

Claim 370 (Proposition X.95: Side of fifth-apotome area produces rational-plus-medial complement)**.** If an area be contained by a rational straight line and a fifth apotome, the side of the area is the line producing with a rational area a medial whole.

Evidence 370 (Proof of X.95)**.** Dual of X.58.

Claim 371 (Proposition X.96: Side of sixth-apotome area produces medial-plus-medial complement)**.** If an area be contained by a rational straight line and a sixth apotome, the side of the area is the line producing with a medial area a medial whole.

Evidence 371 (Proof of X.96)**.** Dual of X.59.

Claim 372 (Proposition X.97: Square on apotome yields first apotome)**.** The square on an apotome straight line applied to a rational straight line produces as breadth a first apotome.

Evidence 372 (Proof of X.97)**.** Inverse of X.91.

Claim 373 (Proposition X.98: Square on first apotome of medial yields second apotome)**.** The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.

Evidence 373 (Proof of X.98)**.** Inverse of X.92.

Claim 374 (Proposition X.99: Square on second apotome of medial yields third apotome)**.** The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.

Evidence 374 (Proof of X.99)**.** Inverse of X.93.

Claim 375 (Proposition X.100: Square on minor yields fourth apotome)**.** The square on a minor applied to a rational straight line produces as breadth a fourth apotome.

Evidence 375 (Proof of X.100)**.** Inverse of X.94.

Claim 376 (Proposition X.101: Square on rational-plus-medial producer yields fifth apotome)**.** The square on the line producing with a rational area a medial whole applied to a rational straight line produces as breadth a fifth apotome.

Evidence 376 (Proof of X.101)**.** Inverse of X.95.

Claim 377 (Proposition X.102: Square on medial-plus-medial producer yields sixth apotome)**.** The square on the line producing with a medial area a medial whole applied to a rational straight line produces as breadth a sixth apotome.

Evidence 377 (Proof of X.102)**.** Inverse of X.96.

Claim 378 (Proposition X.103: Commensurable with apotome is apotome)**.** A straight line commensurable in length with an apotome is itself an apotome and the same in order.

Evidence 378 (Proof of X.103)**.** Dual of X.66.

Claim 379 (Proposition X.104: Commensurable with apotome of medial is the same)**.** A straight line commensurable in length with an apotome of a medial is itself such an apotome of the same order.

Evidence 379 (Proof of X.104)**.** Dual of X.67.

Claim 380 (Proposition X.105: Commensurable with minor is minor)**.** A straight line commensurable with a minor is itself a minor.

Evidence 380 (Proof of X.105)**.** Dual of X.68.

Claim 381 (Proposition X.106: Commensurable with rational-medial producer is the same)**.** A straight line commensurable with the line producing with a rational area a medial whole is itself such a line.

Evidence 381 (Proof of X.106)**.** Dual of X.69.

Claim 382 (Proposition X.107: Commensurable with medial-medial producer is the same)**.** A straight line commensurable with the line producing with a medial area a medial whole is itself such a line.

Evidence 382 (Proof of X.107)**.** Dual of X.70.

Claim 383 (Proposition X.108: Side of rational minus medial is one of the four irrationals)**.** If from a rational area a medial area be subtracted, the side of the remaining area arises as one of four irrationals: an apotome, a first apotome of a medial, a minor, or the line producing with a rational area a medial whole.

Evidence 383 (Proof of X.108)**.** Dual of X.71.

Claim 384 (Proposition X.109: Medial minus rational yields apotome or producer-of-medial)**.** If from a medial area a rational area be subtracted, two other irrational straight lines arise, namely a first apotome of a medial or the line producing with a rational area a medial whole.

Evidence 384 (Proof of X.109)**.** Variant of X.108.

Claim 385 (Proposition X.110: Medial minus medial yields second apotome of medial or medial-producer)**.** If from a medial area there be subtracted a medial area incommensurable with the whole, the remaining two irrational straight lines arise: a second apotome of a medial or the line producing with a medial area a medial whole.

Evidence 385 (Proof of X.110)**.** Variant of X.108 / X.109.

Claim 386 (Proposition X.111: Apotome and binomial are distinct)**.** The apotome is not the same as the binomial.

Evidence 386 (Proof of X.111)**.** A binomial has a rational sum of squares plus a medial rectangle; an apotome has a rational difference of squares minus a medial rectangle; if they coincided, the two combinations would coincide, forcing the medial part to be rational — contradiction.

Claim 387 (Proposition X.112: Square on rational divided by binomial is an apotome)**.** The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and in the same ratio.

Evidence 387 (Proof of X.112)**.** $R^2/(a+b) = a' - b'$ with a', b' in the same ratio as a, b. Verified by direct manipulation of the square-on-binomial identity.

Claim 388 (Proposition X.113: Square on rational divided by apotome is a binomial)**.** The square on a rational straight line applied to an apotome produces as breadth a binomial the terms of which are commensurable with the terms of the apotome and in the same ratio.

Evidence 388 (Proof of X.113)**.** Inverse of X.112.

Claim 389 (Proposition X.114: Rectangle on binomial and apotome can be rational)**.** If an area be contained by an apotome and the binomial the terms of which are commensurable with the terms of the apotome and in the same ratio, the side of the area is rational.

Evidence 389 (Proof of X.114)**.** The rectangle on $(a - b)$ and $(a' + b')$ with $a' = ka$, $b' = kb$ equals $k(a^2 - b^2)$, which is rational.

Claim 390 (Proposition X.115: Medials yield infinitely many irrationals)**.** From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same with any preceding.

Evidence 390 (Proof of X.115)**.** By repeated mean-proportional construction (VI.13) on the medial, each new line is irrational with respect to all earlier ones (using the unique-decomposition results X.42–X.47, X.79–X.84).

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Claim 391 (Proposition XI.1: A straight line cannot have part in a plane and part not)**.** A part of a straight line cannot be in the plane of reference and a part in a plane more elevated.

Evidence 391 (Proof of XI.1)**.** If a straight line AB had part in one plane and continued part in another, then through B there would be two distinct straight lines from A (one in each plane), contradicting Postulate 1 (uniqueness of the straight line through two points).

Claim 392 (Proposition XI.2: Intersecting lines determine a plane)**.** If two straight lines cut one another, they are in one plane, and every triangle is in one plane.

Evidence 392 (Proof of XI.2)**.** Two intersecting lines pick out three non-collinear points (one at the intersection, one on each line); through these three points passes exactly one plane (the analogue in 3D of Postulate 1).

Claim 393 (Proposition XI.3: Intersection of two planes is a straight line)**.** If two planes cut one another, their common section is a straight line.

Evidence 393 (Proof of XI.3)**.** Take two points A, B in the common section. Draw the straight line AB in each plane; by XI.1 each segment of AB lies in its plane, and uniqueness of the line forces both segments to coincide.

Claim 394 (Proposition XI.4: Line perpendicular to two intersecting lines is perpendicular to their plane)**.** If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them.

Evidence 394 (Proof of XI.4)**.** Take any other straight line ℓ through the foot in the plane; ℓ can be expressed as a sum of perpendicular components on the two given lines (by I.46-style decomposition), and the perpendicular to both is perpendicular to the sum by I.4 applied to the right triangles formed.

Claim 395 (Proposition XI.5: Three lines through a point each perpendicular to a fourth lie in a plane)**.** If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane.

Evidence 395 (Proof of XI.5)**.** Two of the three meeting lines determine a plane (XI.2); if the third were out of that plane, the perpendicular relation combined with XI.4 would force two distinct planes through the same set of perpendicular lines, contradiction.

Claim 396 (Proposition XI.6: Two lines perpendicular to the same plane are parallel)**.** If two straight lines be at right angles to the same plane, the straight lines will be parallel.

Evidence 396 (Proof of XI.6)**.** Suppose the two perpendiculars AB, CD met or were skew. Drop the segment BD in the plane; the angles at B and D are right. In the plane through AB and BD, by I.28 the line CD would need to be parallel to AB, fixing the plane through AB, CD. Then within that plane the two right angles force $AB \parallel CD$.

Claim 397 (Proposition XI.7: A line in a plane parallel to a parallel line in another plane lies in the connecting plane)**.** If two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels.

Evidence 397 (Proof of XI.7)**.** Two parallel lines determine a plane (XI.2 extended). Any joining segment between points on the two parallels lies in this plane by XI.1.

Claim 398 (Proposition XI.8: Two parallel lines, one perpendicular to a plane, the other also perpendicular)**.** If two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane.

Evidence 398 (Proof of XI.8)**.** By XI.7 the connecting segment lies in the plane of the parallels; combine XI.4 (perpendicularity transferred along parallel lines via shared plane structure) with I.29 (alternate angles) to obtain perpendicularity of the second line.

Claim 399 (Proposition XI.9: Lines parallel to the same line are parallel)**.** Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another.

Evidence 399 (Proof of XI.9)**.** Construct in each plane perpendiculars from a common point to the shared straight line; the perpendiculars are equal in length, and by I.33 the resulting transversal is parallel to both targets.

Claim 400 (Proposition XI.10: Two angles with parallel sides are equal in space)**.** If two straight lines meeting one another be parallel to two straight lines meeting one another, not in the same plane, they will contain equal angles.

Evidence 400 (Proof of XI.10)**.** Construct the parallelogram joining corresponding points; by I.33 opposite sides are equal, and by I.8 (SSS) the two triangles formed at the vertex angles are congruent.

Claim 401 (Proposition XI.11: Drop a perpendicular from an external point to a plane)**.** From a given elevated point to draw a straight line perpendicular to a given plane.

Evidence 401 (Proof of XI.11)**.** Drop a chord through the point parallel to the plane; drop a perpendicular from the chord to its foot in the plane; the constructed line, being perpendicular to two intersecting lines at the foot, is perpendicular to the plane (XI.4).

Claim 402 (Proposition XI.12: Erect a perpendicular to a plane at a given point in the plane)**.** To set up a straight line at right angles to a given plane from a given point in it.

Evidence 402 (Proof of XI.12)**.** Take an external point above the plane; drop a perpendicular from it to the plane via XI.11; the foot may be made to coincide with the given point by an additional parallel translation (using XI.8).

Claim 403 (Proposition XI.13: A unique perpendicular to a plane from a given point)**.** From the same point two straight lines cannot be set up at right angles to the same plane on the same side.

Evidence 403 (Proof of XI.13)**.** Two such perpendiculars would meet two lines in the plane at the same right angles; by XI.4 / XI.6 the two perpendiculars would have to be parallel; but parallels do not meet – contradicting their common origin.

Claim 404 (Proposition XI.14: Planes perpendicular to the same line are parallel)**.** Planes to which the same straight line is at right angles will be parallel.

Evidence 404 (Proof of XI.14)**.** If the two planes met, the line of intersection (XI.3) would meet the common perpendicular at right angles in two distinct places – contradicting XI.13.

Claim 405 (Proposition XI.15: Two intersecting line-pairs parallel to two other line-pairs give parallel planes)**.** If two straight lines meeting one another be parallel to two straight lines meeting one another, not being in the same plane, the planes through them are parallel.

Evidence 405 (Proof of XI.15)**.** By XI.10 the angles are equal; drop a common perpendicular line; by XI.14 the two planes share a common perpendicular and are parallel.

Claim 406 (Proposition XI.16: A plane cuts parallel planes in parallel lines)**.** If two parallel planes be cut by any plane, their common sections are parallel.

Evidence 406 (Proof of XI.16)**.** The two intersection lines lie in the cutting plane, and if they met, the meeting point would belong to both parallel planes – which is impossible.

Claim 407 (Proposition XI.17: Two parallel planes cut a transversal proportionally)**.** If two straight lines be cut by parallel planes, they will be cut in the same ratios.

Evidence 407 (Proof of XI.17)**.** Draw a parallelogram structure between the parallel planes; apply VI.2 (basic proportionality) in each pair of cross-sectional lines.

Claim 408 (Proposition XI.18: A line perpendicular to a plane makes the plane through it perpendicular to that plane)**.** If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane.

Evidence 408 (Proof of XI.18)**.** Any plane through the perpendicular contains the perpendicular line; by Definition XI.4 (perpendicular planes) the containing plane is perpendicular to the original plane.

Claim 409 (Proposition XI.19: Two perpendicular planes intersect in a line perpendicular to the base)**.** If two planes which cut one another be at right angles to any plane, their common section will also be at right angles to the same plane.

Evidence 409 (Proof of XI.19)**.** The intersection line lies in both planes; by Definition XI.4 the perpendiculars from any point of the intersection within each plane are perpendicular to the base plane; XI.13 then forces the intersection line itself to be perpendicular.

Claim 410 (Proposition XI.20: Solid angle inequality (triangle inequality for face-angles))**.** If a solid angle be contained by three plane angles, any two, taken together in any manner, are greater than the remaining one.

Evidence 410 (Proof of XI.20)**.** Suppose the largest face-angle is $\angle BAC$. Within $\angle BAC$ construct $\angle BAD$ equal to $\angle BAE$ (one of the other face-angles). By I.4 / I.24, the corresponding chord arcs in space give the desired strict triangle-style inequality among the face-angles.

Claim 411 (Proposition XI.21: Sum of face-angles at a solid angle is less than four right angles)**.** Any solid angle is contained by plane angles less than four right angles.

Evidence 411 (Proof of XI.21). Cut a small polygon by a plane near the apex; the sum of the exterior angles of this polygon is less than $4 \cdot 90°$ (by I.32 / I.34 applied to the polygon). The interior face-angles at the apex are the supplements of these exterior angles, so their sum falls strictly short of $4 \cdot 90°$.

Claim 412 (Proposition XI.22: Three face-angles whose sum is less than four right angles can form a solid angle). If there be three plane angles of which two, taken together in any manner, are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines.

Evidence 412 (Proof of XI.22). The plane-angle inequality (XI.20) is precisely the triangle inequality for the joining chords; by I.22 (construction of a triangle on three given segments) the resulting triangle exists.

Claim 413 (Proposition XI.23: Construct a solid angle from three given face-angles). To construct a solid angle out of three plane angles, two of which, taken together in any manner, are greater than the remaining one; thus the sum of the three angles must be less than four right angles.

Evidence 413 (Proof of XI.23). Apply XI.22 to obtain the triangle of chords; mount that triangle so that the three face-angles meet at a common apex; the bound of XI.21 ensures consistency.

Claim 414 (Proposition XI.24: Parallelepiped has parallelogram faces). If a solid be contained by parallel planes, the opposite planes in it are equal and similar parallelograms.

Evidence 414 (Proof of XI.24). Opposite faces share parallel sides (by XI.16) and equal angles (by XI.10), so they are congruent parallelograms by I.33 / I.34.

Claim 415 (Proposition XI.25: Parallelepiped cut by a plane parallel to a face is divided proportionally). If a parallelepipedal solid be cut by a plane parallel to opposite planes, then, as the base is to the base, so will the solid be to the solid.

Evidence 415 (Proof of XI.25). Apply XI.17 to the side faces and VI.1 to the parallel base-pairs; the volume is proportional to one varying side at constant cross-section.

Claim 416 (Proposition XI.26: Construct a solid angle equal to a given solid angle). At a given point on a given straight line to construct a solid angle equal to a given solid angle contained by three plane angles.

Evidence 416 (Proof of XI.26). Reproduce each face-angle by I.23 in the appropriate planes; by XI.23 the resulting figure determines a solid angle congruent to the given one.

Claim 417 (Proposition XI.27: Construct a parallelepiped similar to a given parallelepiped on a given edge). On a given straight line to construct a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid.

Evidence 417 (Proof of XI.27). Apply the face-angle construction of XI.26 at each vertex; by VI.18 the resulting faces are similar to the corresponding faces of the given solid.

Claim 418 (Proposition XI.28: Parallelepiped bisected by its diagonal plane). If a parallelepipedal solid be cut by a plane through the diagonals of the opposite planes, the solid will be bisected by the plane.

Evidence 418 (Proof of XI.28). The two pieces are mirror-image prisms with congruent base-triangles (by I.34); by XI.24 their volumes are equal.

Claim 419 (Proposition XI.29: Parallelepipeds on equal bases with same height are equal)**.** Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are on the same straight lines, are equal to one another.

Evidence 419 (Proof of XI.29)**.** The two parallelepipeds can be decomposed into congruent prisms via XI.28 and I.34 applied repeatedly; the equality is the 3D analogue of I.35 (parallelograms on the same base between the same parallels).

Claim 420 (Proposition XI.30: Parallelepipeds on equal bases with same height but different oblique placement are equal)**.** Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are not on the same straight lines, are equal to one another.

Evidence 420 (Proof of XI.30)**.** Variant of XI.29 with the oblique sides at different angles; the decomposition argument still applies after a shear.

Claim 421 (Proposition XI.31: Parallelepipeds on equal bases are in the ratio of their heights)**.** Parallelepipedal solids which are on equal bases and of the same height are equal to one another.

Evidence 421 (Proof of XI.31)**.** By XI.25 the volume is proportional to the base at fixed height; if the bases are equal, the volumes are equal.

Claim 422 (Proposition XI.32: Parallelepipeds of equal height are as their bases)**.** Parallelepipedal solids which are of the same height are to one another as their bases.

Evidence 422 (Proof of XI.32)**.** By XI.25 applied with one shared dimension fixed.

Claim 423 (Proposition XI.33: Similar parallelepipeds are in the triplicate ratio of corresponding edges)**.** Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

Evidence 423 (Proof of XI.33)**.** The 3D analogue of VI.20: scaling each of the three edges by ratio k multiplies the volume by k^3. Apply VI.20 to two faces and XI.32 to extrude.

Claim 424 (Proposition XI.34: Equal parallelepipeds have reciprocally proportional edges)**.** In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

Evidence 424 (Proof of XI.34)**.** 3D analogue of VI.14: by XI.32 the ratio of volumes is the compounded ratio of bases and heights; equal volumes force the compounded ratio to be unity, i.e. reciprocal proportion.

Claim 425 (Proposition XI.35: Planes equally inclined to a base have equal perpendiculars)**.** If there be two equal plane angles, and on their vertices there be set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points be taken at random and perpendiculars be drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines be joined to the vertices of the original angles, they will contain, with the elevated straight lines, equal angles.

Evidence 425 (Proof of XI.35)**.** By I.4 (SAS) applied to the right triangles in each plane: equal oblique segments and equal perpendiculars produce equal angles at the foot.

Claim 426 (Proposition XI.36: Parallelepiped on three proportionals is equal to a cube whose side is the mean)**.** If three straight lines be proportional, the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.

Evidence 426 (Proof of XI.36)**.** For $a : b = b : c$ (with b the mean), $abc = b^3$ and the parallelepipeds on $\{a, b, c\}$ versus $\{b, b, b\}$ have equal volumes by XI.34 / XI.33.

Claim 427 (Proposition XI.37: Similar parallelepipeds in proportion)**.** If four straight lines be proportional, the similar and similarly described parallelepipedal solids upon them will also be proportional; and if the similar and similarly described parallelepipedal solids upon them be proportional, the straight lines will themselves also be proportional.

Evidence 427 (Proof of XI.37)**.** 3D analogue of VI.22: similar parallelepipeds are in the triplicate ratio of edges; equality of triplicate ratios is equivalent to equality of edge ratios.

Claim 428 (Proposition XI.38: Joining diagonals of opposite faces in a cube)**.** If the sides of the opposite planes of a cube be bisected, and planes be carried through the points of section, the common section of the planes and the diameter of the cube bisect one another.

Evidence 428 (Proof of XI.38)**.** By symmetry of the cube under the half-turns about the centre; the diagonal and the median plane both pass through the centre.

Claim 429 (Proposition XI.39: Prisms on equal triangular bases with same height are equal)**.** If there be two prisms of equal height, and one have a parallelogram as base and the other a triangle, and if the parallelogram be double of the triangle, the prisms will be equal.

Evidence 429 (Proof of XI.39)**.** A triangular prism is half a parallelogram prism on the same height (by I.34 applied to the cross-sections); the area-doubling condition makes the two prisms have equal volume.

12 Book XII — Method of Exhaustion

Claim 430 (Proposition XII.1: Similar inscribed polygons in circles are as squares on diameters)**.** Similar polygons inscribed in circles are to one another as the squares on the diameters.

Evidence 430 (Proof of XII.1)**.** Decompose each polygon into similar triangles by joining vertices to the centres; each pair of corresponding triangles is similar (VI.20) with side ratio equal to the diameter ratio; sum and combine via V.12.

Claim 431 (Proposition XII.2: Circles are as squares on diameters)**.** Circles are to one another as the squares on the diameters.

Evidence 431 (Proof of XII.2)**.** Apply the method of exhaustion: inscribe similar polygons in the two circles; by XII.1 they are in the ratio of squares on the diameters. Any deviation from that ratio at the level of the circles leads, via X.1, to a contradiction by choosing inscribed polygons close enough to fill the circle.

Claim 432 (Proposition XII.3: A pyramid on a triangular base is split into two equal smaller pyramids and two equal prisms)**.** Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole, and having triangular bases, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

Evidence 432 (Proof of XII.3)**.** Bisect each edge (I.10); the midpoint cuts split the pyramid into two corner pyramids and two prisms. The two corner pyramids are similar to the original (their edges halved), and by I.34 the two prisms are congruent.

Claim 433 (Proposition XII.4: Pyramids of equal height are as their bases (special case))**.** If there be two pyramids of the same height which have triangular bases, and each of them be divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then, as the base of the one pyramid is to the base of the other pyramid, so will all the prisms in the one pyramid be to all the prisms in the other pyramid.

Evidence 433 (Proof of XII.4)**.** Each iteration of XII.3 doubles the number of prisms; by proportionality of bases (VI.1) the prism-sums maintain the same ratio as the original bases.

Claim 434 (Proposition XII.5: Pyramids of equal height are as their bases)**.** Pyramids which are of the same height and have triangular bases are to one another as their bases.

Evidence 434 (Proof of XII.5)**.** Apply XII.4 in the limit of XII.3 iterations; the prism-sums exhaust the pyramids (X.1), so the base-ratio is the pyramid-ratio.

Claim 435 (Proposition XII.6: Pyramids of equal height on polygonal bases are as their bases)**.** Pyramids which are of the same height and have polygonal bases are to one another as the bases.

Evidence 435 (Proof of XII.6)**.** Triangulate each polygonal base; the polygon-pyramid is the sum of the triangular sub-pyramids; apply XII.5 and V.12.

Claim 436 (Proposition XII.7: Triangular prism is three equal pyramids)**.** Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.

Evidence 436 (Proof of XII.7)**.** Cut the prism by two planes through opposite edge-pairs; the three resulting pyramids share a common apex and have congruent base triangles, so by XII.5 they have equal volume.

Claim 437 (Proposition XII.8: Similar pyramids are as the cubes on corresponding edges)**.** Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides.

Evidence 437 (Proof of XII.8)**.** By XII.7 a prism is three equal pyramids; by XI.33 similar parallelepipeds (and hence prisms) are in the triplicate ratio of edges; transfer to pyramids by XII.5.

Claim 438 (Proposition XII.9: Equal pyramids have reciprocally proportional bases and heights)**.** In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights; and those pyramids which have triangular bases in which the bases are reciprocally proportional to the heights are equal.

Evidence 438 (Proof of XII.9)**.** 3D analogue of VI.15 for pyramids; via XII.5 / XII.6 the area-times- height proportion factors into the reciprocal proportion of bases and heights.

Claim 439 (Proposition XII.10: A cone is one-third of the cylinder on the same base and height)**.** Any cone is a third part of the cylinder which has the same base with it and equal height.

Evidence 439 (Proof of XII.10)**.** Inscribe a pyramid on a polygonal base in both cone and cylinder; XII.7 makes the pyramid one-third the prism; apply X.1 to refine the inscribed polygon to fill the circle (XII.2); the limit gives the cone-to-cylinder ratio.

Claim 440 (Proposition XII.11: Cones and cylinders of equal height are as their bases)**.** Cones and cylinders which are of the same height are to one another as their bases.

Evidence 440 (Proof of XII.11)**.** By XII.2 the bases (circles) are in the squared-diameter ratio; by the formula in XII.10, the volumes follow the same ratio.

Claim 441 (Proposition XII.12: Similar cones and cylinders are as cubes on diameters)**.** Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases.

Evidence 441 (Proof of XII.12)**.** Analogue of XII.8 for cones / cylinders: by similarity the height scales proportionally with the diameter; cube of the linear ratio gives the volume ratio.

Claim 442 (Proposition XII.13: Parallel sections of a cylinder are in ratio of distances)**.** If a cylinder be cut by a plane which is parallel to its opposite planes, then, as the cylinder is to the cylinder, so will the axis be to the axis.

Evidence 442 (Proof of XII.13)**.** Parallel cross-sections give equal circles (XI.16 implies parallel diameters); the volume scales linearly with axial length by XII.11.

Claim 443 (Proposition XII.14: Cylinders on equal bases are as their heights)**.** Cones and cylinders which are on equal bases are to one another as their heights.

Evidence 443 (Proof of XII.14)**.** By XII.13 the volume is proportional to the axis when the base is fixed.

Claim 444 (Proposition XII.15: Equal cones / cylinders have reciprocal proportions)**.** In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.

Evidence 444 (Proof of XII.15)**.** Analogue of XII.9 / VI.15 for cones and cylinders.

Claim 445 (Proposition XII.16: Inscribe in the larger of two concentric circles a polygon not touching the smaller)**.** Given two circles about the same centre, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle.

Evidence 445 (Proof of XII.16)**.** Bisect arcs repeatedly (III.30) until the chord-to-arc gap is smaller than the difference of radii; this guarantees that the inscribed polygon avoids touching the smaller circle.

Claim 446 (Proposition XII.17: Inscribe in the larger of two concentric spheres a polyhedron not touching the smaller)**.** Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Evidence 446 (Proof of XII.17)**.** 3D analogue of XII.16: apply XII.16 in great-circle cross-sections, then triangulate the sphere using XI.27 to assemble a polyhedron strictly inside the larger sphere and outside the smaller.

Claim 447 (Proposition XII.18: Spheres are in triplicate ratio of diameters)**.** Spheres are to one another in the triplicate ratio of their respective diameters.

Evidence 447 (Proof of XII.18)**.** Apply the method of exhaustion: inscribe similar polyhedra (XII.17); by XII.12 (similar cones) and the polyhedron's similar-pyramid decomposition, the inscribed solids are in the triplicate ratio of diameters. By X.1 the inscribed solids approach the spheres in volume; the limit gives the result.

13 Book XIII — Platonic Solids

Claim 448 (Proposition XIII.1: Square on the whole plus square on half segment)**.** If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square on the half.

Evidence 448 (Proof of XIII.1)**.** Let AB be cut at C in extreme and mean ratio with $AC > CB$. Let D be the midpoint of AB. Apply II.6: $(AB/2 + AC)^2 = (AB/2)^2 + AB \cdot AC + AC^2$. By the defining relation $AC^2 = AB \cdot CB$, simplification gives $(AB/2 + AC)^2 = 5(AB/2)^2$.

Claim 449 (Proposition XIII.2: Square on segment plus square on smaller part)**.** If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

Evidence 449 (Proof of XIII.2)**.** Converse of XIII.1: assume the squared relation and deduce the extreme-and-mean cut using II.6 / II.11.

Claim 450 (Proposition XIII.3: Squares on extreme-and-mean parts)**.** If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment.

Evidence 450 (Proof of XIII.3)**.** Apply II.6 / II.11 to the lesser-segment configuration; algebraic analogue of XIII.1.

Claim 451 (Proposition XIII.4: Squares on whole, greater, and lesser are commensurable)**.** If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.

Evidence 451 (Proof of XIII.4)**.** $AB^2+CB^2 = 3 \cdot AC^2$ where C cuts AB in extreme-and-mean ratio (greater AC). Use $AC^2 = AB \cdot CB$ and II.4 to verify the identity.

Claim 452 (Proposition XIII.5: Extension preserves extreme-and-mean property)**.** If a straight line be cut in extreme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line is cut in extreme and mean ratio, and the original straight line is the greater segment.

Evidence 452 (Proof of XIII.5)**.** Extending by the greater segment AC to A' (so $A'A = AC$, AB the original), check that $A'A : AB = AB : (A'A+AB-AC)$, which reduces via the original extreme-and-mean relation to the same form.

Claim 453 (Proposition XIII.6: Greater segment of a rational divided in extreme-and-mean is an apotome)**.** If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome.

Evidence 453 (Proof of XIII.6)**.** Solve $x^2 + ax = a^2$ for the greater segment $x = a(\sqrt{5} - 1)/2$; this is the form of an apotome relative to the rational a (Book X classification).

Claim 454 (Proposition XIII.7: Three angles of equilateral pentagon equal implies all equal)**.** If three angles of an equilateral pentagon, taken either in order or not in order, be equal, the pentagon will be equiangular.

Evidence 454 (Proof of XIII.7)**.** The five interior angles sum to $3 \cdot 180°$ (I.32 extended); combined with three equal angles, the constraint forces all five to be equal.

Claim 455 (Proposition XIII.8: Diagonals of a regular pentagon cut each other in extreme-and-mean)**.** If in an equilateral and equiangular pentagon straight lines subtend two adjacent angles, they cut one another in extreme and mean ratio, and the greater segments are equal to the side of the pentagon.

Evidence 455 (Proof of XIII.8)**.** Construct the pentagon inscribed in a circle (IV.11). Two diagonals form an isosceles triangle with vertex angle 36° (I.32 / IV.10); by similarity (VI.4) the diagonal-segment ratio matches the extreme-and-mean ratio.

Claim 456 (Proposition XIII.9: Hexagon side plus decagon side in extreme-and-mean)**.** If the side of the hexagon and that of the decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

Evidence 456 (Proof of XIII.9)**.** The hexagon side equals the radius (IV.15); the decagon side satisfies the 36-72-72 triangle relations (IV.10); their sum is in the golden ratio to the hexagon side.

Claim 457 (Proposition XIII.10: Pentagon side squared equals hexagon plus decagon sides squared)**.** If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle.

Evidence 457 (Proof of XIII.10)**.** This is the Pythagorean relation $p^2 = h^2 + d^2$ in the inscribed polygons of a unit circle. Proven via I.47 applied to the right triangle formed by the centre, a pentagon-vertex, and a decagon-vertex.

Claim 458 (Proposition XIII.11: Side of inscribed pentagon in rational circle is minor irrational)**.** If in a circle which has its diameter rational an equilateral pentagon be inscribed, the side of the pentagon is the irrational straight line called minor.

Evidence 458 (Proof of XIII.11)**.** By XIII.10 the pentagon side is $\sqrt{h^2 + d^2}$ with h rational and d an apotome (XIII.6); the resulting form falls in Book X's minor class (Definition XIII.4 / X.76).

Claim 459 (Proposition XIII.12: Side of inscribed equilateral triangle squared is three times square on radius)**.** If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius.

Evidence 459 (Proof of XIII.12)**.** For inscribed equilateral triangle, side $= r\sqrt{3}$. Proven via I.47 on the perpendicular bisector configuration.

Claim 460 (Proposition XIII.13: Construct a regular tetrahedron in a sphere)**.** To construct a pyramid (regular tetrahedron), to comprehend it in a given sphere, and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Evidence 460 (Proof of XIII.13)**.** Inscribe an equilateral triangle (IV.2); erect an apex above the centroid at height $r\sqrt{2/3}$ where r is the circumradius. The four equal edges form the tetrahedron; place the sphere through its four vertices. The diameter-squared / side-squared $= 3/2$.

Claim 461 (Proposition XIII.14: Construct a regular octahedron in a sphere)**.** To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.

Evidence 461 (Proof of XIII.14)**.** Take two perpendicular diameters in a circle; through the centre erect a perpendicular axis equal in length to the diameter. The four endpoints in the circle and two endpoints on the axis form the six vertices of the octahedron. Diameter-squared / side-squared $= 2$.

Claim 462 (Proposition XIII.15: Construct a cube in a sphere)**.** To construct a cube and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

Evidence 462 (Proof of XIII.15)**.** Take a square base (IV.6); erect a parallel square at height equal to the side. The eight vertices form the cube; the sphere through them has diameter $\sqrt{3}$ times the side.

Claim 463 (Proposition XIII.16: Construct a regular icosahedron in a sphere)**.** To construct an icosahedron and comprehend it in a sphere, as in the case of the aforesaid figures; and to prove that the side of the icosahedron is the irrational straight line called minor.

Evidence 463 (Proof of XIII.16)**.** Inscribe a regular pentagon in a circle (IV.11); arrange two parallel pentagons rotated 36° from each other, plus two apex points. Twelve vertices form the icosahedron. The side is a minor straight line by XIII.11.

Claim 464 (Proposition XIII.17: Construct a regular dodecahedron in a sphere)**.** To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the side of the dodecahedron is the irrational straight line called apotome.

Evidence 464 (Proof of XIII.17)**.** The dodecahedron has twelve regular pentagonal faces; the side is the apotome formed when the cube-edge is cut in extreme and mean ratio (XIII.6). Inscribe by placing pentagonal faces on the six square faces of the inscribed cube (XIII.15).

Claim 465 (Proposition XIII.18: There are exactly five regular solids)**.** To set out the sides of the five figures and to compare them with one another; and that no other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another.

Evidence 465 (Proof of XIII.18)**.** Compare the side lengths: tetrahedron $\sqrt{2/3}$, octahedron $\sqrt{1/2}$, cube $1/\sqrt{3}$, icosahedron (minor irrational), dodecahedron (apotome). For the uniqueness clause: at each vertex of a regular polyhedron, the sum of face-angles must be less than four right angles (XI.21). Equilateral triangles (60°): 3, 4, or 5 around a vertex — tetrahedron, octahedron, icosahedron. Squares (90°): only 3 around a vertex — cube. Regular pentagons (108°): only 3 around a vertex — dodecahedron. Hexagons (120°): three would tile flat, no vertex — impossible. Larger polygons: even three exceed 360°. Hence exactly five regular polyhedra exist.